Chapter 5

Supersymmetric Quantum Mechanics on $\mathbb{CP}^2$

Appendix 5(i)

$\mathbb{CP}^n$

Complex projective n-dimensional space is defined as flat (n+1)-dimensional complex space minus the origin, with points identified that are equal up to a scaling by a complex number

$$\mathbb{CP}^n = \frac{\{[z] \neq 0 \in \mathbb{C}^{n+1}\}}{[z] \sim [\lambda z]}$$

$\mathbb{CP}^n$ can be covered by n+1 coordinate patches. In terms of the n+1 homogeneous coordinates $[Z_0, Z_1, \cdots, Z_n]$, the simplest way to do this is to take inhomogeneous coordinates

$$[\frac{Z_1}{Z_0}, \frac{Z_2}{Z_0}, \cdots, \frac{Z_n}{Z_0}] \text{ everywhere except } Z_0 = 0 \qquad ,$$

$$[\frac{Z_0}{Z_1}, \frac{Z_2}{Z_1}, \cdots, \frac{Z_n}{Z_1}] \text{ everywhere except } Z_1 = 0 \qquad ,$$

and similarly for each patch up to the (n+1)-th

$$[\frac{Z_0}{Z_n}, \frac{Z_1}{Z_n}, \cdots, \frac{Z_{n-1}}{Z_n}] \text{ everywhere except } Z_n = 0 \qquad .$$

Coodinates transformations on the overlap of these patches are manifestly holomorphic, so $\mathbb{CP}^n$ are complex manifolds. Letting $\{w_i = \frac{Z_i}{Z_0}\}$ be coordinates in the first coordinate patch $(Z_0 \neq 0)$, a metric may be put on $\mathbb{CP}^n$. This metric, the Fubini-Study metric takes the form

$$ds^2 = \frac{\sum_i dw_i d\bar{w}_i}{1 + \sum_i w_i \bar{w}_i} - \frac{\sum_{i,j} \bar{w}_i w_j dw_i d\bar{w}_i}{(1 + \sum_i w_i \bar{w}_i)^2}$$

The associated Kähler form is closed, so $\mathbb{CP}^n$ are Kähler manifolds. They are in fact the Kähler manifolds with minimal cohomology, the only harmonic (p,q)-forms being exterior powers of the Kähler form.

The Hodge Numbers are therefore

$$h^{p,p}(\mathbb{CP}^n) = 1 \qquad , \qquad 0 \leq p \leq n \qquad ,$$
$$h^{p,q}(\mathbb{CP}^n) = 0 \qquad , \qquad p \neq q \qquad ,$$

which gives the Euler Characteristic as

$$\chi(\mathbb{CP}^n) = n + 1 \qquad .$$

$\mathbb{CP}^n$ can be shown to be equivalent to the homogeneous space

$$\mathbb{CP}^n = \frac{SU(n+1)}{SU(n) \times U(1)}$$

where $SU(n+1)$ is the isometry group of the $\mathbb{CP}^n$, i.e. the group of diffeomorphisms of the manifold which leaves the metric invariant, $U(n) = SU(n) \times U(1)$ is the isotropy or stability group, the subgroup of the isometry group which leaves a particular point fixed, the $U(1)$ factor is due to the freedom to scale by a complex number.

$\mathbb{CP}^1$ is equivalent to the 2-sphere.

5. Supersymmetric Quantum Mechanics on $\mathbb{CP}^n$

The second illustration of the relationship between supersymmetric quantum mechanics and fixed point theorems is with the manifold $\mathbb{CP}^2$. This is an example where the fixed point set of a Killing Vector need not just consist of isolated fixed points.

Firstly the ordinary Laplacian on $\mathbb{CP}^2$ is dealt with and the Frobenius Reciprocity Theorem is used to demonstrate that all the solutions have been found. Killing Vectors are then introduced into the supersymmetry algebra, the zero energy solutions are related to the topology of the manifold and perturbation theory is used to show how the symmetry breaking affects the excited states.

5.1 $\mathbb{CP}^2$

$\mathbb{CP}^2$ can be represented by three homogeneous complex coordinates (Z_0, Z_1, Z_2), with points identified that are equal up to scaling by a complex number. In terms of the inhomogeneous coordinates

$$z = \frac{Z_1}{Z_0}, u = \frac{Z_2}{Z_0} \qquad ,$$

the Fubini-Study metric tensor on $\mathbb{CP}^2$ takes the form

$$g_{a\bar{b}} = (1 + z\bar{z} + u\bar{u})^{-2}) \begin{pmatrix} 1 + u\bar{u} & -u\bar{z} \\ -\bar{u}z & 1 + z\bar{z} \end{pmatrix}$$

and so the Kähler form is

$$\omega = (1 + z\bar{z} + u\bar{u})^{-2}[(1 + u\bar{u})dz \wedge d\bar{z} \ + (1 + z\bar{z})du \wedge d\bar{u} - u\bar{z}dz \wedge d\bar{u} - \bar{u}zdu \wedge d\bar{z}] \quad .$$

The isometry group of $\mathbb{CP}^2$ is SU(3). Explicit formulae for the generators of this group can be found by taking the generators of U(3), the isometry group of $\mathbb{C}^3$, and transforming from the homogeneous coordinates on $\mathbb{CP}^2$ to the inhomogeneous coordinates. The nine generators of U(3) are

$$Z_i \frac{\partial}{\partial Z_j} - \bar{Z}_j \frac{\partial}{\partial \bar{Z}_i} \qquad , \qquad \text{where } i, j = 0, 1, 2 \, .$$

Transforming each of these generators into inhomogeneous coordinates gives

$$Z_1 \frac{\partial}{\partial Z_1} - \bar{Z}_1 \frac{\partial}{\partial \bar{Z}_1} = z\frac{\partial}{\partial z} - \bar{z}\frac{\partial}{\partial \bar{z}} = \frac{2}{3}(I_3 + V_3)$$

$$Z_1 \frac{\partial}{\partial Z_2} - \bar{Z}_2 \frac{\partial}{\partial \bar{Z}_1} = z\frac{\partial}{\partial u} - \bar{u}\frac{\partial}{\partial \bar{z}} = \sqrt{2}V_+$$

$$Z_1 \frac{\partial}{\partial Z_0} - \bar{Z}_0 \frac{\partial}{\partial \bar{Z}_1} = -z^2\frac{\partial}{\partial z} - uz\frac{\partial}{\partial u} - \frac{\partial}{\partial \bar{z}} = \sqrt{2}I_+$$

$$Z_2 \frac{\partial}{\partial Z_1} - \bar{Z}_1 \frac{\partial}{\partial \bar{Z}_2} = -\bar{z}\frac{\partial}{\partial \bar{u}} + u\frac{\partial}{\partial z} = \sqrt{2}V_-$$

$$Z_2 \frac{\partial}{\partial Z_2} - \bar{Z}_2 \frac{\partial}{\partial \bar{Z}_2} = u\frac{\partial}{\partial u} - \bar{u}\frac{\partial}{\partial \bar{u}} = \frac{2}{3}(U_3 + V_3)$$

$$Z_2\frac{\partial}{\partial Z_0} - \bar{Z}_0\frac{\partial}{\partial \bar{Z}_2} = -u^2\frac{\partial}{\partial u} - uz\frac{\partial}{\partial z} - \frac{\partial}{\partial \bar{u}} = \sqrt{2}U_-$$

$$Z_0\frac{\partial}{\partial Z_1} - \bar{Z}_1\frac{\partial}{\partial \bar{Z}_0} = \bar{z}^2\frac{\partial}{\partial \bar{z}} + \bar{u}\bar{z}\frac{\partial}{\partial \bar{u}} + \frac{\partial}{\partial z} = \sqrt{2}I_-$$

$$Z_0\frac{\partial}{\partial Z_2} - \bar{Z}_2\frac{\partial}{\partial \bar{Z}_0} = \bar{u}^2\frac{\partial}{\partial \bar{u}} + \bar{u}\bar{z}\frac{\partial}{\partial \bar{z}} + \frac{\partial}{\partial u} = \sqrt{2}U_+$$

$$Z_0\frac{\partial}{\partial Z_0} - \bar{Z}_0\frac{\partial}{\partial \bar{Z}_0} = -[z\frac{\partial}{\partial z} - \bar{z}\frac{\partial}{\partial \bar{z}} + u\frac{\partial}{\partial u} - \bar{u}\frac{\partial}{\partial \bar{u}}] = \frac{2}{3}(U_3 - I_3) \qquad .$$

Thus in terms of the inhomogeneous coordinates there are eight independent operators, the generators of SU(3). [The fifth operator minus the first is equal to the last operator on the list.] A U(1) factor having been lost due to the freedom to scale the homogeneous coordinates. We will take as a basis for SU(3):

The Cartan Sub-Algebra

$$I_3 = z\frac{\partial}{\partial z} - \bar{z}\frac{\partial}{\partial \bar{z}} - \frac{1}{2}[\bar{u}\frac{\partial}{\partial \bar{u}} - u\frac{\partial}{\partial u}]$$

$$U_3 = \bar{u}\frac{\partial}{\partial \bar{u}} - u\frac{\partial}{\partial u} - \frac{1}{2}[z\frac{\partial}{\partial z} - \bar{z}\frac{\partial}{\partial \bar{z}}]$$

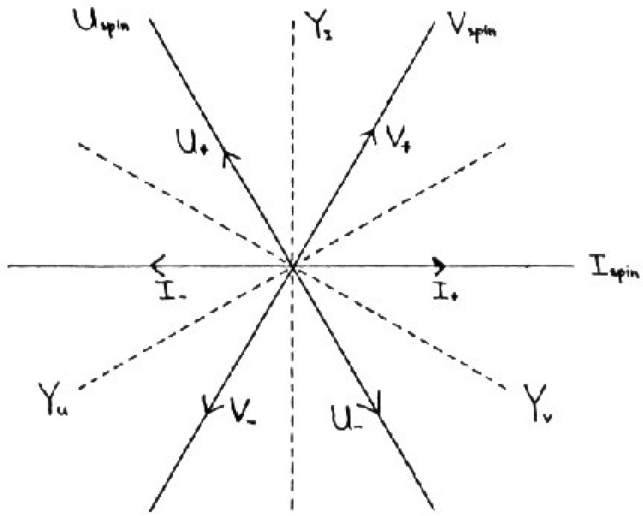
and the raising and lowering operators

$$I_- = \frac{1}{\sqrt{2}}[\bar{z}^2\frac{\partial}{\partial \bar{z}} + \bar{u}\bar{z}\frac{\partial}{\partial \bar{u}} + \frac{\partial}{\partial z}] \qquad , \qquad I_+ = -\bar{I}_-$$

$$U_- = \frac{-1}{\sqrt{2}}[u^2\frac{\partial}{\partial u} + uz\frac{\partial}{\partial z} + \frac{\partial}{\partial \bar{u}}] \qquad , \qquad U_+ = -\bar{U}_-$$

$$V_- = \frac{1}{\sqrt{2}}[u\frac{\partial}{\partial z} - \bar{z}\frac{\partial}{\partial \bar{u}}] \qquad , \qquad V_+ = -\bar{V}_- \qquad .$$

In terms of an SU(3) root diagram the basis takes the following form:



where I_3 measures the third componenet of the SU(2) subgroup I_{spin} , the distance of a weight along the I_{spin} axis, and U_3 measures the corresponding distance along the U_{spin} axis. Y_I, Y_U, Y_V are hypercharge operators, which commute with I_{spin}, U_{spin} and V_{spin} respectively.

$$Y_I = \bar{u}\frac{\partial}{\partial \bar{u}} - u\frac{\partial}{\partial u}$$

$$Y_U = z\frac{\partial}{\partial z} - \bar{z}\frac{\partial}{\partial \bar{z}}$$

$$Y_V = [z\frac{\partial}{\partial z} - \bar{z}\frac{\partial}{\partial \bar{z}}] - [\bar{u}\frac{\partial}{\partial \bar{u}} - u\frac{\partial}{\partial u}]$$

5.2 Supersymmetry on $\mathbb{CP}^2$

Defining, as before, the supersymmetry operators

$$Q_1 = \partial + \partial^* \quad , \qquad Q_2 = \bar{\partial} + \bar{\partial}^* \quad , \qquad Q_3 = i(\partial - \partial^*) \qquad , \qquad Q_4 = i(\bar{\partial} - \bar{\partial}^*)$$

gives the supersymmetry algebra

$$H = Q_i^2 = \frac{1}{2}\triangle_d \qquad , \qquad \{Q_i, Q_j\} = 0 \ , \qquad i \neq j \qquad .$$

Due to the invariance of the metric under the isometry group the eigensolutions of the Laplacian on $\mathbb{CP}^2$ form representations of SU(3), but only certain representations of SU(3) occur. To determine the complete set of eigensolutions it is necessary to employ the Frobenius Reciprocity Theorem.

5.2.1 The Frobenius Reciprocity Theorem [19][20]

$\mathbb{CP}^n$ are examples of homogeneous spaces G/H , where G is the isometry group and H is the isotropy group, the subgroup which leaves one point fixed. The eigensolutions of the Laplacian on a homogeneous space form the representations of the isometry group G induced by some representation of the isotropy group H.

The Frobenius Reciprocity Theorem determines how induced representations decompose into irreducible representations. The theorem states that the multiplicity of the occurrence of a particular representation of G, in the representation induced by a representation R of H, is equal to the multiplicity of R in the irreducible representation of G when it is decomposed under the subgroup H. In the case in question G and H are infinite groups, so the induced representation is an infinite dimensional function space and decomposes into an infinite number of irreducible representations of G.

Taking as an example the two-sphere $S^2 \equiv \mathbb{CP}^1 \equiv SU(2)/U(1)$ the zero-form eigensolutions of the Laplacian transform trivially under the isotropy group and so form the representation of SU(2) induced by the trivial representation of U(1). To decompose this representation each irreducible representation of SU(2) is examined to see how many times the trivial representation of U(1), i.e. the chargeless singlet, occurs after decomposition under U(1). It is straightforward to see that the odd-dimensional representations of SU(2), $\mathbf{1}, \mathbf{3}, \mathbf{5}, \cdots$ each contain one chargeless singlet when decomposed under U(1) and the even-dimensional representations $\mathbf{2}, \mathbf{4}, \mathbf{6}, \cdots$ don't contain any. The Frobenius Reciprocity Theorem therefore implies that the eigensolutions of the Laplacian form one copy each of the odd-dimensional irreducible representations of SU(2). These are in fact the Associated Legendre Funcions $Y_n^m(z)$. This is the reason why orbital angular momentum is never half integer valued.

One-forms transform as a covector. Under the isotropy group U(1) this two-dimensional covector decomposes into a (1,)-form with charge +1 and a (0,1)-form with charge -1. These form two separate irreducible representations of U(1). Both the +1 and the -1 representations occur once in the decomposition under U(1) of each of the odd-dimensional representations of SU(2) apart from the singlet, i.e. $\mathbf{3}, \mathbf{5}, \mathbf{7}, \cdots$, but not in the even-dimensional representations. Therefore the one-form eigensolutions of the Laplacian on S^2 form two copies of each of the odd-dimensional representations of SU(2) apart from the singlet, one copy consisting of (1,0)-forms, the other copy being (0,1)-forms.

The two-form eigensolutions are Poincare dual to zero-form eigensolutions and so form the same SU(2) representations, that is, one copy of the odd-dimensional representations $\mathbf{1}, \mathbf{3}, \mathbf{5}, \cdots$.

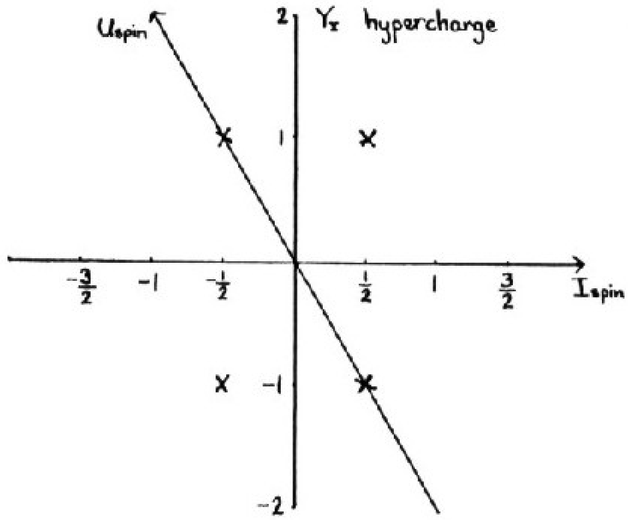
When the Laplacian on S^2 was considered in the previous chapter the Frobenius Reciprocity Theorem was not essential to understand which representations were formed by the eigensolutions. It is well known that the scalar spherical harmonics are the Associated Legendre Functions and the one-form and two-form eigensolutions must all be related to these by supersymmetry, forming supersymmetry quadruplets.

On $\mathbb{CP}^2$ things are more complicated and without invoking the Frobenius Reciprocity Theorem it is not obvious which irreducible representations of SU(3) occur as eigensolutions of the Laplacian.

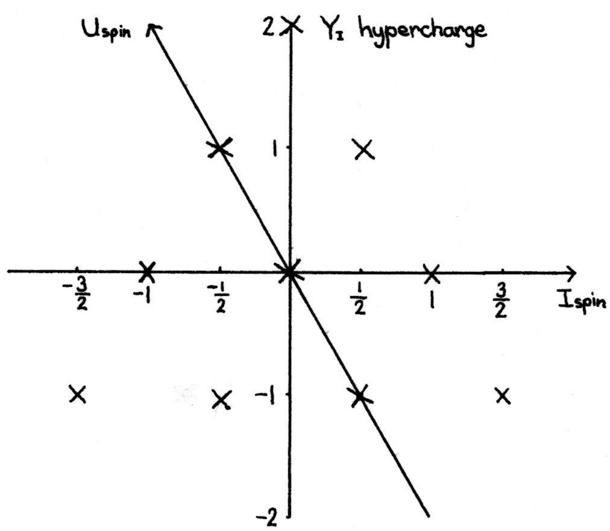
$\mathbb{CP}^2$ is equivalent to the homogeneous space $SU(3)/SU(2)$, so zero-form eigensolutions of the Laplacian on $\mathbb{CP}^2$ form the representation of SU(3) induced by the trivial representation of U(2). The multiplicity of an irreducible representation of SU(3) in this representation is equal to the number of hypercharge zero singlets it contains after decomposition under U(2).

Moving inwards in the weight diagram of an irreducible SU(3) representation, the number of weights at each point in a particular layer increases by one every time the previous layer was hexagonal. Once a triangular layer is reached the number of weights at each point in this triangle and within are the same. Decomposing an irreducible representation of SU(3) under U(2), a hypercharge zero singlet will only be obtained if the number of weights at each point increases in each layer all the way to the centre. In this case the hypercharge zero weights will decomposed into the U(2) representations $\mathbf{1}^0 \oplus \mathbf{3}^0 \oplus \mathbf{5}^0 \oplus \cdots$ Thus a hypercharge zero singlet occurs once in the decomposition of the regular hexagon representations, but not in the decomposition of any other irreducible representations. Therefore the only representations occurring as zero-form eigensolutions of the Laplacian on $\mathbb{CP}^2$ are the trivial representation and the regular hexagons, and each of these occurs once. The dimension of the n-th regular hexagon representation of SU(3) is $(n+1)^3$.

One-forms transform as covectors which on $\mathbb{CP}^2$ form the following four-dimensional representation on an SU(3) weight diagram (where hypercharge has been scaled by a factor of $\frac{2}{\sqrt{3}}$).



and decompose under U(2) into two doublets, a (1,0)-form doublet 2^1 and a (0,1)-form doublet 2^{-1}.
After decomposing the irreducible representations of SU(3) under U(2) it can be seen that the regular hexagons all contain both doublets, the hypercharge +1 states decomposing into $\mathbf{2}^1 \oplus \mathbf{4}^1 \oplus \mathbf{6}^1 \oplus \cdots$ and the hypercharge -1 states decomposing into $\mathbf{2}^{-1} \oplus \mathbf{4}^{-1} \oplus \mathbf{6}^{-1} \oplus \cdots$.
The hypercharge +1 doublet is also contained in the decomposition of the triangular decuplet representation

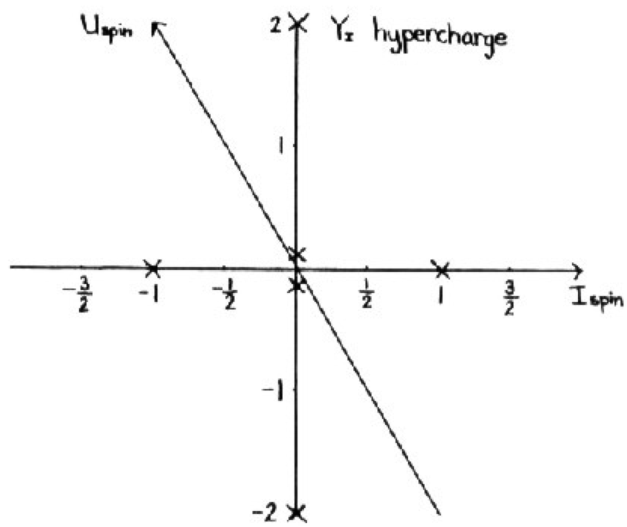


and a sequence of larger representations produced by surrounding the decuplet with irregular hexagons. If the decuplet is considered as an irregular hexagon with sides of length nought and three, the next representation is an irregular hexagon with sides of length one and four, the 35-dimensional representation, the next is an irregular hexagon with sides of length two and six, the 81-dimensional representation, and so on.

The hypercharge -1 doublet is contained in the decomposition of the complex conjugate representations, $\bar{\mathbf{10}}$, the $\bar{\mathbf{35}}$, the $\bar{\mathbf{81}}$ etc.

The one-form eigensolutions of the Laplacian on $\mathbb{CP}^2$ therefore consist of a (1,0)-form copy of each of the regular hexagonal representations, a (1,0)-form copy of each of the sequence of irregular hexagonal representations which contains the triangular decuplet at the centre including the decuplet itself, a (0,1)-form copy of each of the regular hexagons and a (0,1)-form copy of the complex conjugate of the previous irregular hexagonal representations.

Two-forms transform as anti-symmetric second rank cotensors. Taking anti-symmetric combinations of two of the previous four-dimensional covector representations gives the following six-dimensional representation on an SU(3) weight diagram,



which decomposes under U(2) as a (2,0)-form singlet $\mathbf{1^2}$, a (1,1)-form triplet $\mathbf{3^0}$, a (1,1)-form singlet $\mathbf{1^0}$ and a (0,2)-form singlet $\mathbf{1^{-2}}$.

The hypercharge zero singlet $\mathbf{1^0}$ is contained in the decomposition of the SU(3) singlet and each of the regular hexagons. The hypercharge zero triplet $\mathbf{3^0}$ is contained once in each of the regular hexagons, once in the decomposition of the decuplet and its sequence of irregular hexagons and also once in the complex conjugate of the decuplet and the complex conjugate of the sequence of irregular hexagons. The $\mathbf{1^2}$ representation is contained once in the decomposition of the decuplet and each of its sequence of irregular hexagons and the $\mathbf{1^{-2}}$ representation is contained once in the decomposition of each of the complex conjugate representations.

The two-form eigensolutions of the Laplacian on $\mathbb{CP}^2$ therefore consist of a (1,1)-form SU(3) singlet, two (1,1)-form copies of each of the regular hexagon representations, a (2,0)-form and a (1,1)-form copy of the decuplet and each of its sequence of irregular hexagons and a (0,2)-form and a (1,1)-form copy of the anti-decuplet and each of its sequence of irregular hexagons.

The three-form eigensolutions correspond to the one-form eigensolutions in a one-to-one way, as do the four-form eigensolutions to the zero-form eigensolutions due to Poincare duality.

Taking these results as a whole it can be seen that the Frobenius Reciprocity Theorem indicates that on $\mathbb{CP}^2$ the Laplacian has three singlet solutions, a (0,0)-form, a (1,1)-form and a (2,2)-form. These three solutions must have zero energy as there are no one-form or three-form singlet solutions to which they could be mapped by the supersymmetry operators, so the Frobenius Reciprocity Theorem is powerful enough to actually determine the cohomology of the manifold. The three zero energy solutions are in fact the constant (0,0)-form, the Kähler form ω and the volume form V.

$$V = \frac{1}{2}\omega \wedge \omega = \frac{1}{(1 + z\bar{z} + u\bar{u})^3} dz \wedge du \wedge d\bar{z} \wedge d\bar{u}$$

The rest of the solutions form supersymmetry quadruplets. Each of the regular hexagons comes in two quadruplets, one containing a (0,0)-form, a (1,0)-form, a (0,1)-form and a (1,1)-form and a Poincare dual quadruplet containing a (2,2)-form, a (1,2)-form, (2,1)-form and a (1,1)-form. Each of the decuplet and its sequence of irregular hexagons comes in one quadruplet containing a (1,0)-form, a (2,0)-form, a (1,1)-form and a (2,1)-form. Each of the anti-decuplet and its sequence of irregular hexagons comes in a quadruplet consisting of a (0,1)-form, a (1,1)-form, a (0,2)-form and a (1,2)-form.

All the states contained in the regular hexagon representations may be found by considering the Laplacian on scalars and using supersymmetry operators and Poincare duality. Similarly all the states contained in the irregular hexagon representations may be found by considering the Laplacian on one-forms.

5.2.2 The Regular Hexagon Representations

The Laplacian on scalars is

$$\triangle_\partial = -g^{a\bar{b}} \frac{\partial^2}{\partial \xi^a \partial \bar{\xi^b}} \qquad , \qquad \xi^1 = z, \xi^2 = u$$

$$= -(1 + z\bar{z} + u\bar{u})[(1 + z\bar{z})\frac{\partial^2}{\partial z \partial \bar{z}} + (1 + u\bar{u})\frac{\partial^2}{\partial u \partial \bar{u}} + \bar{u}z\frac{\partial^2}{\partial \bar{u} \partial z} + u\bar{z}\frac{\partial^2}{\partial \bar{z} \partial \bar{u}}]$$

Eigensolutions of the Laplacian which are members of regular hexagons will be denoted $\chi_n^{i,j,(k)}(z,u)$, n signifies the level of the representation, i and j are the eigenvalues of the operators I_3 and U_3 respectively, and (k) is a degenergy label for states occupying the same point in the SU(3) weight diagram. The highest weight eigensolutions $\chi_n^{n,n}(z,u)$ take the form

$$\chi_n^{n,n}(z,u) = \frac{\bar{u}^n z^n}{(1 + z\bar{z} + u\bar{u})^n}$$

It is quite straightforward to compute the action of the Laplacian on these functions.

$$\frac{d\chi_n^{n,n}(z,u)}{dz} = \frac{n\bar{u}^n z^{n-1}}{(1+z\bar{z}+u\bar{u})^n} - \frac{n\bar{u}^n z^n \bar{z}}{(1+z\bar{z}+u\bar{u})^{n+1}}$$

$$= \frac{n\bar{u}^n z^{n-1}(1+u\bar{u})}{(1+z\bar{z}+u\bar{u})^{n+1}}$$

$$\frac{d\chi_n^{n,n}(z,u)}{du} = \frac{-n\bar{u}^{n+1} z^n}{(1+z\bar{z}+u\bar{u})^{n+1}}$$

$$\frac{d^2\chi_n^{n,n}(z,u)}{dzd\bar{z}} = \frac{-n(n+1)\bar{u}^n z^n(1+u\bar{u})}{(1+z\bar{z}+u\bar{u})^{n+2}}$$

$$\frac{d^2\chi_n^{n,n}(z,u)}{dzd\bar{u}} = \frac{n^2\bar{u}^{n-1} z^{n-1}}{(1+z\bar{z}+u\bar{u})^{n+1}} + \frac{n(n+1)\bar{u}^n z^{n-1} u}{(1+z\bar{z}+u\bar{u})^{n+1}} - \frac{n(n+1)\bar{u}^n z^{n-1} u(1+u\bar{u})}{(1+z\bar{z}+u\bar{u})^{n+2}}$$

$$= \frac{n^2\bar{u}^{n-1} z^{n-1}}{(1+z\bar{z}+u\bar{u})^{n+1}} + \frac{n(n+1)\bar{u}^n z^{n-1} u z\bar{z}}{(1+z\bar{z}+u\bar{u})^{n+2}}$$

$$\frac{d^2\chi_n^{n,n}(z,u)}{d\bar{z}du} = \frac{n(n+1)\bar{u}^{n+1} z^{n+1}}{(1+z\bar{z}+u\bar{u})^{n+2}}$$

$$\frac{d^2\chi_n^{n,n}(z,u)}{dud\bar{u}} = \frac{-n(n+1)\bar{u}^n z^n(1+z\bar{z})}{(1+z\bar{z}+u\bar{u})^{n+2}}$$

which gives finally

$$\triangle_\partial \chi_n^{n,n}(z,u) = \frac{1}{(1+z\bar{z}+u\bar{u})}[2n(n+1)(1+u\bar{u})(1+z\bar{z}) - n^2(1+z\bar{z}+u\bar{u}) - 2n(n+1)u\bar{u}z\bar{z}]\chi_n^{n,n}(z,u)$$

$$= \frac{1}{(1+z\bar{z}+u\bar{u})}[(2n(n+1)-n^2)(1+u\bar{u}+z\bar{z})]\chi_n^{n,n}(z,u)$$

$$= n(n+2)\chi_n^{n,n}(z,u) \qquad .$$

It is again straightforward to show that this state is indeed a highest weight state, and so is annihilated by I_+ and U_+ .

$$-\sqrt{2}I_+ = z^2\frac{\partial}{\partial z} + uz\frac{\partial}{\partial u} + \frac{\partial}{\partial \bar{z}}$$

$$-\sqrt{2}I_+\chi_n^{n,n}(z,u)=\frac{n\bar{u}^n z^{n+1}(1+u\bar{u})}{(1+z\bar{z}+u\bar{u})^{n+1}}-\frac{nu\bar{u}^{n+1}z^{n+1}}{(1+z\bar{z}+u\bar{u})^{n+1}}-\frac{n\bar{u}^n z^{n+1}}{(1+z\bar{z}+u\bar{u})^{n+1}}=0 \quad .$$

Under the transformation $\bar{u}\longleftrightarrow z$, the raising operators are transformed into each other $I_+\longleftrightarrow -U_+$. The state $\chi_n^{n,n}(z,u)$ is symmetrical under this operation and so must be annihilated by U_+ as well as I_+ .

Thus the representation at level n has dimension $(n+1)^3$ and energy $E=n(n+2)$, and all the states in the representation may be found by applying the lowering operators; I_- , U_- and V_- to the highest weight state $\chi_n^{n,n}(z,u)$.

5.2.3 The First Regular Hexagon States

The first excited state form the adjoint representation of SU(3), the octet, and have energy E = 3.

$$\frac{\bar{u}}{1+z\bar{z}+u\bar{u}} \qquad\qquad \frac{\bar{u}z}{1+z\bar{z}+u\bar{u}}$$

$$\frac{\bar{z}}{1+z\bar{z}+u\bar{u}} \qquad \begin{matrix}\dfrac{1-z\bar{z}}{1+z\bar{z}+u\bar{u}}\\[2ex] \dfrac{1-u\bar{u}}{1+z\bar{z}+u\bar{u}}\end{matrix} \qquad \frac{z}{1+z\bar{z}+u\bar{u}}$$

$$\frac{\bar{u}z}{1+z\bar{z}+u\bar{u}} \qquad\qquad \frac{\bar{u}}{1+z\bar{z}+u\bar{u}}$$

The second excited states obtainable from the highest weight state $\chi_2^{2,2}(z,u)=\dfrac{\bar{u}^2z^2}{(1+z\bar{z}+u\bar{u})^2}$ form the **27** representation. All these states are functions with denominators $(1+z\bar{z}+u\bar{u})^2$. The numerators are

$$\bar{u}^2 \qquad \bar{u}^2z \qquad \bar{u}^2z^2$$

$$\bar{u}\bar{z} \qquad \begin{matrix}\bar{u}(1-u\bar{u})\\ \bar{u}(1-2z\bar{z})\end{matrix} \qquad \begin{matrix}\bar{u}z(2-u\bar{u})\\ \bar{u}z(2-z\bar{z})\end{matrix} \qquad \bar{u}z^2$$

$$\bar{z}^2 \qquad \begin{matrix}\bar{z}(1-z\bar{z})\\ \bar{z}(1-2u\bar{u})\end{matrix} \qquad \begin{matrix}3(1-u\bar{u})^2-(1+u\bar{u})^2\\ 3(1-z\bar{z})^2-(1+z\bar{z})^2\\ 3(u\bar{u}-z\bar{z})^2-(u\bar{u}+z\bar{z})^2\end{matrix} \qquad \begin{matrix}z(1-z\bar{z})\\ z(1-2u\bar{u})\end{matrix} \qquad z^2$$

$$u\bar{z}^2 \qquad \begin{matrix}u\bar{z}(2-u\bar{u})\\ u\bar{z}(2-z\bar{z})\end{matrix} \qquad \begin{matrix}u(1-u\bar{u})\\ u(1-2z\bar{z})\end{matrix} \qquad uz$$

$$u^2\bar{z}^2 \qquad u^2\bar{z} \qquad u^2$$

and the energy of the states is $E=8$.

5.2.4 Irregular Hexagon Representations

The other sequence of representations which form eigensolutions of the Laplacian on $\mathbb{CP}^2$ may be found by consideration of the Laplacian acting on (1,0)-forms. The (1,0)-forms

$$\psi_n^{\frac{1-n}{2},n+1}(z,u) = \frac{\bar{u}^{n+1}dz}{(1+z\bar{z}+u\bar{u})^{n+1}} \ , \qquad n \geq 1$$

are states in the n-th irregular hexagon representations. n must be greater than or equal to one for the state to be normalizable. These states are annihilated by the Lie Derivative along the ladder operators, $\mathcal{L}_{\mathbf{U}_+}, \mathcal{L}_{\mathbf{V}_+}, \mathcal{L}_{\mathbf{I}_-}$ and so are outer weight states.

The holomorphic exterior derivative and its adjoint act on these one-forms thus

$$^*\psi_n^{\frac{1-n}{2},n+1}(z,u) = -\frac{u^{n+1}(1+z\bar{z})}{(1+z\bar{z}+u\bar{u})^{n+3}}du\wedge d\bar{u}\wedge d\bar{z} + \frac{\bar{z}u^{n+2}}{(1+z\bar{z}+u\bar{u})^{n+3}}dz\wedge d\bar{u}\wedge d\bar{z}$$

$$\partial^*\psi_n^{\frac{1-n}{2},n+1}(z,u) = 0$$

so $\psi_n^{\frac{1-n}{2},n+1}(z,u)$ are coclosed with respect to the holomorphic exterior derivative.

$$\partial\psi_n^{\frac{1-n}{2},n+1}(z,u) = \frac{-(n+1)\bar{u}^{n+2}}{(1+z\bar{z}+u\bar{u})^{n+2}}du\wedge dz$$

$$^*\partial\psi_n^{\frac{1-n}{2},n+1}(z,u) = \frac{-(n+1)u^{n+2}}{(1+z\bar{z}+u\bar{u})^{n+2}}d\bar{u}\wedge d\bar{z}$$

$$\partial^*\partial\psi_n^{\frac{1-n}{2},n+1}(z,u) = (n+1)(n+2)[\frac{-u^{n+1}(1+z\bar{z})}{(1+z\bar{z}+u\bar{u})^{n+3}}du\wedge d\bar{u}\wedge d\bar{z} + \frac{\bar{z}u^{n+2}}{(1+z\bar{z}+u\bar{u})^{n+3}}dz\wedge d\bar{u}\wedge d\bar{z}]$$

$$= (n+1)(n+2)^*\psi_n^{\frac{1-n}{2},n+1}(z,u)$$

which gives finally for the operation of the Laplacian

$$\triangle\psi_n^{\frac{1-n}{2},n+1}(z,u) = {}^*\partial^*\partial\psi_n^{\frac{1-n}{2},n+1}(z,u) = (n+1)(n+2)\psi_n^{\frac{1-n}{2},n+1}(z,u)$$

so the energy of the n-th irregular hexagon representation is $E = (n+1)(n+2)$.

The other members of the representations may be found by acting on these outer weight states with the Lie Derivatives along the ladder operators: $\mathcal{L}_{\mathbf{U}_+}, \mathcal{L}_{\mathbf{V}_+}, \mathcal{L}_{\mathbf{I}_-}$. The dimension of the n-th lowest of these representations is given by the formula

$$\Sigma_n = \frac{n}{2}(n+3)(2n+3)$$

the first one being the decuplet. The (1,0)-forms which make up the decuplet representation and have energy $E = 6$ all have denominators $(1 + u\bar{u} + z\bar{z})^2$ with numerators as follows

$$\bar{u}^2 dz$$

$$-2\bar{u}\bar{z}dz + \bar{u}^2 du \qquad \bar{u}(2 - u\bar{u})dz + \bar{u}^2 z du$$

$$\bar{z}^2 dz - 2\bar{u}\bar{z}du \qquad (1 - u\bar{u})\bar{z}dz - (1 - z\bar{z})\bar{u}du \qquad (1 - 2u\bar{u})dz + 2z\bar{u}du$$

$$\bar{z}^2 du \qquad u\bar{z}^2 dz + \bar{z}(2 - z\bar{z})du \qquad 2\bar{z}udz + (1 - 2z\bar{z})du \qquad zdu - udz$$

These representations are complex so their complex conjugates must also be (0,1)-form eigensolutions of the Laplacian, the $\bar{\mathbf{10}}$, the $\bar{\mathbf{35}}$, the $\bar{\mathbf{81}}$ etc.

The other three members of the supersymmetry quadruplets containing these 1-forms; i.e. the two 2-forms and a 3-form may of course be found by applying the supersymmetry operators.

5.3 Killing Vectors on $\mathbb{CP}^2$

The fact that SU(3) is a rank 2 Lie group means that it is possible to introduce a two parameter family of Killing Vectors into the supersymmetry algebra. The simplest way to do this is in terms of the two hypercharge operators Y_I and Y_U

$$\mathbf{k} = is[z\frac{\partial}{\partial z} - \bar{z}\frac{\partial}{\partial \bar{z}}] + it[u\frac{\partial}{\partial u} - \bar{u}\frac{\partial}{\partial \bar{u}}]$$

The effect of the Killing Vector acting on the coordinates is an infinitesimal rotation, which generically will leave three points fixed. The easiest way to find the fixed points is to consider the homogeneous coordinates (Z_0, Z_1, Z_2). In these coordinates the Killing Vector takes the form

$$\mathbf{k} = is[Z_1\frac{\partial}{\partial Z_1} - \bar{Z}_1\frac{\partial}{\partial \bar{Z}_1}] + it[Z_2\frac{\partial}{\partial Z_2} - \bar{Z}_2\frac{\partial}{\partial \bar{Z}_2}]$$

and the effect of this operator on the homogeneous coordinates is

$$(Z_0, Z_1, Z_2) \longrightarrow (Z_0, Z_1 + i\varepsilon s Z_1, Z_2 + i\varepsilon t Z_2)$$

where ε is an infinitesimal parameter, so the only points unchanged up to a scaling by a complex number are

$$x_1 = (1,0,0) \ , \qquad x_2 = (0,1,0) \ , \qquad x_3 = (0,0,1) \ ,$$

the three fixed points.

In terms of the inhomogeneous coordinates these three points are the origins in the three coordinate patches used to cover $\mathbb{CP}^2$. In the first coordinate patch

$$[z = \frac{Z_1}{Z_0}, u = \frac{Z_2}{Z_0}] \qquad , \qquad Z_0 \neq 0$$

the origin $(z = 0, u = 0)$ corresponds to $x_1 = (1,0,0)$.

In the second coordinate patch

$$[x = \frac{Z_0}{Z_1}, y = \frac{Z_2}{Z_1}] \qquad , \qquad Z_1 \neq 0$$

which in terms of the coordinates (z, u) on the overlap of the patches $(x = \frac{1}{z}, y = \frac{u}{z})$. Using the chain rule to transform the coordinates, the Killing Vector in the second patch take the form

$$\mathbf{k} = -is[x\frac{\partial}{\partial x} - \bar{x}\frac{\partial}{\partial \bar{x}}] + i(t-s)[y\frac{\partial}{\partial y} - \bar{y}\frac{\partial}{\partial \bar{y}}] \qquad .$$

The fixed point in this patch is again the origin $(x = 0, y = 0)$ and corresponds to the point $x_2 = (0,1,0)$.
In the third coordinate patch

$$[w = \frac{Z_0}{Z_2}, v = \frac{Z_1}{Z_2}] \qquad , \qquad Z_2 \neq 0$$

the Killing Vector takes the form

$$\mathbf{k} = -it[w\frac{\partial}{\partial w} - \bar{w}\frac{\partial}{\partial \bar{w}}] + i(s-t)[v\frac{\partial}{\partial v} - \bar{v}\frac{\partial}{\partial \bar{v}}] \qquad .$$

The fixed point is the origin $(w = 0, v = 0)$ and corresponds to the point $x_3 = (0,0,1)$.
There are also three cases when the fixed point set of the Killing Vector doesn't just consist of isolated fixed points, but is composed of one fixed point plus a fixed $\mathbb{CP}^1$ submanifold. The three cases for which this occurs are when s = 0 or t = 0 or s = t. In these cases the Killing Vector is an SU(3) hypercharge operator and so commutes with an SU(2) subgroup of SU(3), this SU(2) being the isometry group of the fixed $\mathbb{CP}^1$ submanifold. The first of these Killing Vectors in terms of the homogeneous coordinates is

$$\mathbf{k} = is[Z_1\frac{\partial}{\partial Z_1} - \bar{Z}_1\frac{\partial}{\partial \bar{Z}_1}]$$

and its infinitesimal action on the coordinates is

$$(Z_0, Z_1, Z_2) \longrightarrow (Z_0, Z_1 + i\varepsilon s Z_1, Z_2)$$

so the fixed point set consists of the point

$$x_2 = (0,1,0)$$

and fixed $\mathbb{CP}^1$ submanifold

$$S^2_{(2)} = (Z_0, 0, Z_2) \qquad .$$

In terms of the inhomogeneous coordinates, the first and third patches on $\mathbb{CP}^2$ are needed to completely cover this fixed $\mathbb{CP}^1$ submanifold. The form of the Killing Vector in these two patches is the same

$$\mathbf{k} = is[z\frac{\partial}{\partial z} - \bar{z}\frac{\partial}{\partial \bar{z}}]$$

in the first patch, and

$$\mathbf{k} = is[v\frac{\partial}{\partial v} - \bar{v}\frac{\partial}{\partial \bar{v}}]$$

in the third patch. The inhomogeneous coordinates that correspond to the fixed $\mathbb{CP}^1$ are

$(z = 0, u)$ in the first patch,

$(w = \frac{1}{u}, v = \frac{z}{u} = 0)$ in the third patch.

The second hypercharge operator

$$\mathbf{k} = it[Z_2\frac{\partial}{\partial Z_2} - \bar{Z}_2\frac{\partial}{\partial \bar{Z}_2}]$$

has a fixed point set consisting of the fixed point

$$x_3 = (0, 0, 1)$$

and the fixed $\mathbb{CP}^1$ submanifold

$$S^2_{(3)} = (Z_0, Z_1, 0) \qquad .$$

The action of the third hypercharge operator

$$\mathbf{k} = is[Z_1\frac{\partial}{\partial Z_1} - \bar{Z}_1\frac{\partial}{\partial \bar{Z}_1}] + is[Z_2\frac{\partial}{\partial Z_2} - \bar{Z}_2\frac{\partial}{\partial \bar{Z}_2}]$$

on the homogeneous coordinates is

$$(Z_0, Z_1, Z_2) \longrightarrow (Z_0, Z_1 + i\varepsilon s Z_1, Z_2 + i\varepsilon s Z_2)$$

so the fixed point set consists of the fixed point

$$x_1 = (1, 0, 0)$$

and the fixed $\mathbb{CP}^1$ submanifold

$$S^2_{(1)} = (0, Z_1, Z_2) \qquad .$$

The standard form of the Killing Vector on any patch is

$$\mathbf{k} = i\lambda_1[\zeta_1 \frac{\partial}{\partial \zeta_1} - \bar{\zeta}_1 \frac{\partial}{\partial \bar{\zeta}_1}] + i\lambda_2[\zeta_2 \frac{\partial}{\partial \zeta_2} - \bar{\zeta}_2 \frac{\partial}{\partial \bar{\zeta}_2}]$$

where both parameters λ_1 and λ_2 are positive. To achieve this form the coordinates may have to be redefined by interchanging one or both of the holomorphic coordinates with their conjugate coordinates. Each coordinate interchange produces a reversal of the orientation of the patch.

5.4 The Introduction of a Killing Vector into the supersymmetry algebra

The exterior derivative may now be generalized by the inclusion of a Killing Vector to give $d_s + i_{\mathbf{k}}$, or more specifically

$$d_s = d + i_{is(z\frac{\partial}{\partial z} - \bar{z}\frac{\partial}{\partial \bar{z}}) + it(u\frac{\partial}{\partial u} - \bar{u}\frac{\partial}{\partial \bar{u}})}$$

leading to the Hamiltonian H_s as in section 3.

This operator may be split into a holomorphic piece ∂_s and an anti-holomorphic piece $\bar{\partial}_s$, where

$$\bar{\partial}_s = \bar{\partial} + i_{isz\frac{\partial}{\partial z} + itu\frac{\partial}{\partial u}}$$
$$\partial_s = \partial - i_{is\bar{z}\frac{\partial}{\partial \bar{z}} + it\bar{u}\frac{\partial}{\partial \bar{u}}}$$

$\bar{\partial}_s$ maps (p,q)-forms to (p,q+1)-forms and (p,q-1)-forms.

∂_s maps (p,q)-forms to (p=1,q)-forms and (p-1,q)-forms.

5.4.1 Cohomology corresponding to d_s

It has not been possible to find all the harmonic forms in the sense of H_s on $\mathbb{CP}^2$. The independent harmonic forms are however in a one-to-one correspondence with the classes of closed but not exact forms, so it was decided to settle for finding representations of these classes in the sense of d_s . When acting on the subspace of zero energy states $d_s^2 = 0$, so d_s may be treated like a nilpotent operator. Witten ([1] p.28, 29) gives a formula for σ^i , closed but not exact forms in the sense of d_s , where i is an index running over the cohomology classes. With $\mathbb{CP}^2$ being a 4-dimensional manifold this reduces to

$$\sigma^i = \psi^i \wedge (\phi(K^2) + \phi'(K^2)d\tilde{k} + \frac{1}{2}\phi''(K^2)d\tilde{k} \wedge d\tilde{k})$$

where $\phi(K^2)$ is the test function

$$\phi(K^2) = \exp[\frac{-1}{\alpha - K^2}] \quad , \quad K^2 < \alpha$$
$$\phi(K^2) = 0 \quad \text{otherwise}$$
$$0 < \alpha \leq \text{ local max. } (K^2)$$

which is only non-zero over a region of $\mathbb{CP}^2$ containing one fixed submanifold i.e. either a fixed point or a fixed $\mathbb{CP}^1$. This is a continuous function because when the denominator of the exponent's argument is zero it

vanishes along with all its derivatives. The parameter α must therefore take a positive value less than the local maximum of K^2 surrounding the fixed submanifold. ψ^i are the representatives of the cohomology of the fixed submanifold. The dual of the Killing Vector $\tilde{k}$ and the function K^2 are defined in Appendix 3(i).
The ψ^i are annihilated by the exterior derivative d by definition and also by i_k because they are defined on a fixed manifold, therefore $d_s\psi^i = 0$. The fact that the other piece of the σ^i is annihilated by d_s depends on the relation $d(K^2) = -i_k(d\tilde{k})$ which may easily be shown to be true as follows:

$$\mathcal{L}_k k = \mathcal{L}_{[k,k]} = 0$$

$$\mathcal{L}_k g = 0 \qquad \text{because k is a Killing Vector}$$

$$\implies \mathcal{L}_k \tilde{k} = 0$$

or $$(di_k + i_k d)\tilde{k} = 0$$

$$di_k\tilde{k} = -i_k d\tilde{k}$$

$$dK^2 = -i_k d\tilde{k}\ .$$

The non-exactness of the σ^i follows from consideration of the fixed manifolds. On a fixed manifold d_s equals the ordinary exterior derivative d which maps p-forms to (p+1)-forms, and the lowest-form part of σ^i is then $\exp[-\frac{1}{\alpha}]\psi_i$. If σ^i were exact in the sense of d_s this would imply that the ψ^i were exact in the sense of d which is not the case. The other two terms in σ^i on a fixed manifold are manifestly exact as $\phi'(0)$ and $\phi''(0)$ are constants.
All this may be clearly illustrated for both the fixed point and the fixed $\mathbb{CP}^1$ cases by consideration of the Killing Vector k $= is[z\frac{\partial}{\partial z} - \bar{z}\frac{\partial}{\partial \bar{z}}]$. The function K^2 corresponding to this Killing Vector is

$$K^2 = \frac{s^2(1+u\bar{u})z\bar{z}}{(1+z\bar{z}+u\bar{u})^2}$$

This is a degenerate Morse Function with three extrema; the minimum $K^2 = 0$ on the fixed $\mathbb{CP}^1(z=0,u) \oplus (\frac{z}{u}=0, \frac{1}{u}=0)$, the maximum $K^2 = \frac{s^2}{4}$ on the 3-sphere $(z\bar{z} = 1+u\bar{u})$ and the minimum $K^2 = 0$ on the fixed point $(x = \frac{1}{z} = 0, y = \frac{u}{z} = 0)$. The Morse Indices of these critical manifolds are p = 0 for the $\mathbb{CP}^1$ and the point and p = 1 for the S^3. This incidentally gives the following values for the Morse Numbers using the formula at the end of section 1.2.1.

$$M_0 = B_0(\mathbb{CP}^1) + B_0(point) = 1+1 = 2$$

$$M_1 = B_0(S^3) + B_1(\mathbb{CP}^1) = 1+0 = 1$$

$$M_2 = B_1(S^3) + B_2(\mathbb{CP}^1) = 0+1 = 1$$

$$M_3 = B_2(S^3) \qquad = 0$$
$$M_4 = B_3(S^3) \qquad = 1$$

Thus giving the value of the Euler Characteristic of $\mathbb{CP}^2$ as

$$\chi(\mathbb{CP}^2) = \sum_p (-1)^p M_p = 2 - 1 + 1 - 0 + 1 = 3 \quad .$$

Near the fixed $\mathbb{CP}^1$ the two form $d\tilde{k}$ may be calculated to be

$$d\tilde{k} = \frac{-s}{(1+z\bar{z}+u\bar{u})^3}[(1+u\bar{u}-z\bar{z})[(1+u\bar{u})dz\wedge d\bar{z} - z\bar{u}du\wedge d\bar{z} - u\bar{z}dz\wedge d\bar{u}] - (1-u\bar{u}+z\bar{z})z\bar{z}du\wedge d\bar{u}]$$

The two representatives of the cohomology of the $\mathbb{CP}^1$ are

$$\psi^1 = 1 \qquad , \qquad \psi_2 = \frac{du\wedge d\bar{u}}{(1+u\bar{u})^2} \qquad .$$

The parameter α may take any value greater than zero, but less than $\frac{s^2}{4}$, the maximum value of K^2 which is on the S^3. Taking α equal to $\alpha_{max} = \frac{s^2}{4}$ the test function simplifies to give

$$\phi(K^2) = \exp\left(-\left[\frac{2}{s}\frac{(1+u\bar{u}+z\bar{z})}{(1+u\bar{u}-z\bar{z})}\right]^2\right) \qquad , \qquad z\bar{z} < 1+u\bar{u} \quad ,$$

$$\phi(K^2) = 0 \qquad , \qquad z\bar{z} \geq 1+u\bar{u} \quad .$$

Putting all these terms together in Witten's formula gives two even-forms σ^1 and σ^2 closed but not exact in the sense of d_s .

In the second patch the Killing Vector takes the form $\mathbf{k} = -is[x\frac{\partial}{\partial x} - \bar{x}\frac{\partial}{\partial \bar{x}}] - is[y\frac{\partial}{\partial y} - \bar{y}\frac{\partial}{\partial \bar{y}}]$. Near the fixed point, the origin of this patch, K^2 and $d\tilde{k}$ take the following form

$$K^2 = \frac{s^2(x\bar{x}+y\bar{y})}{(1+x\bar{x}+y\bar{y})^2}$$

$$d\tilde{k} = \frac{-s}{(1+x\bar{x}+y\bar{y})^2}\Big[(1+y\bar{y}-x\bar{x})dx\wedge d\bar{x} + (1+x\bar{x}-y\bar{y})dy\wedge d\bar{y} - 2x\bar{y}dy\wedge d\bar{x} - 2\bar{x}ydx\wedge d\bar{y}\Big]$$

The representative of the cohomology is the constant zero-form $\psi = 1$. The nearest maximum to the fixed point is where $x\bar{x} + y\bar{y} = 1$, which is of course the same 3-sphere as previously, where $K^2 = \dfrac{s^2}{4}$. Taking this maximum value for α the test function simplifies to

$$\phi(K^2) = \exp\left(-\left[\frac{2}{s}\frac{(1+x\bar{x}+y\bar{y})}{(1-x\bar{x}-y\bar{y})}\right]^2\right) \quad , \quad x\bar{x}+y\bar{y} < 1 \quad ,$$

$$\phi(K^2) = 0 \quad , \quad x\bar{x}+y\bar{y} \geq 1 \quad .$$

These expressions may be combined therefore to form σ^3 , the third representative of the cohomology of $\mathbb{CP}^2$ in the sense of $d_s = d + i_{\mathbf{k}}$. The σ^i are only defined up to the addition of an exact form, so they are members of a three parameter family which is apparent in the freedom to scale α between zero and $\dfrac{s^2}{4}$. σ^1, σ^2 and σ^3 are all even-forms so the Euler Characteristic of $\mathbb{CP}^2$ is just three, the sum of the dimensions of the even cohomology groups.

5.4.2 The Zero Energy Solutions

Zero energy eigensolutions of the Laplacian corresponding to d_s face much more stringent requirements allowing no freedom to scale any arbitrary parameters. It has been possible to find the two self-dual even-form zero energy eigensolutions, but unfortunately not the anti-self-dual zero energy solution.

Using the anti-holomorphic part of d_s it is possible to show, as follows, that the even-forms

$$\Phi_1^o(z,u) = \exp\left[\frac{-sz\bar{z}-tu\bar{u}}{1+u\bar{u}+z\bar{z}}\right](1-\omega+\frac{1}{2}\omega\wedge\omega)$$

$$\Phi_2^o(z,u) = \exp\left[\frac{sz\bar{z}+tu\bar{u}}{1+u\bar{u}+z\bar{z}}\right](1+\omega+\frac{1}{2}\omega\wedge\omega)$$

where ω is the Kähler form, are zero energy eigensolutions of the Hamiltonian.

$$\bar{\partial}\left[\frac{-sz\bar{z}-tu\bar{u}}{1+u\bar{u}+z\bar{z}}\right] = -s\left[\frac{z(1+u\bar{u})d\bar{z}-uz\bar{z}d\bar{u}}{(1+u\bar{u}+z\bar{z})^2}\right] - t\left[\frac{u(1+z\bar{z})d\bar{u}-zu\bar{u}d\bar{z}}{(1+u\bar{u}+z\bar{z})^2}\right]$$

$$i_{isz\frac{\partial}{\partial z}+itu\frac{\partial}{\partial u}}\omega = \frac{-1}{(1+u\bar{u}+z\bar{z})^2}\left[sz(1+u\bar{u})d\bar{z}+t(1+z\bar{z})ud\bar{u}-suz\bar{z}d\bar{u}-tzu\bar{u}d\bar{z}\right]$$

$$= \bar{\partial}\left[\frac{-sz\bar{z}-tu\bar{u}}{1+u\bar{u}+z\bar{z}}\right]$$

The anti-holomorphic exterior derivative annihilates the Kähler form and the interior product only acts on the exterior algebra, the full operator $\bar{\partial}_s$ therefore annihilates the states $\Phi_1^o(z,u)$ and $\Phi_2^o(z,u)$. Moreover these states are real and so are annihilated by the holomorphic operator ∂_s as well and therefore are closed in terms of the generalized exterior derivative d_s and are therefore zero energy eigensolutions of the Hamiltonian which is the generalized Laplacian.

5.4.3 Excited States

The self-dual zero energy eigensolutions of the Hamiltonian, after the introduction of the Killing Vector into the supersymmetry algebra consist of combinations of the representatives of the cohomology of $\mathbb{CP}^2$ multiplied by scalar functions. These combinations of the representatives of the cohomology may be substituted into the Schrödinger Equation to diagonalize the Hamiltonian. When this has been done the Hamiltonian takes a much simpler form. On scalars it consists of the scalar Laplacian plus a function of z and u.

Taking the first of the zero energy eigensolutions

$$\Phi_1^o(z,u) = \exp\left[\frac{-sz\bar{z} - tu\bar{u}}{1 + u\bar{u} + z\bar{z}}\right](1 - \omega + \frac{1}{2}\omega \wedge \omega) \quad .$$

After the substitution of the differential form part of this solution, $(1 - \omega + \frac{1}{2}\omega \wedge \omega)$, into the Schrödinger Equation the resulting diagonalized Hamiltonian on scalars

$$H_1 = -g^{a\bar{b}}\frac{\partial^2}{\partial\xi^a\partial\bar{\xi^b}} + f_1(z,u)$$

must give zero when acting on the scalar part of the solution, $\exp\left[\frac{-sz\bar{z} - tu\bar{u}}{1 + u\bar{u} + z\bar{z}}\right]$. The action of the Laplacian on this function may therefore be used to find $f_1(z,u)$.

$$-g^{a\bar{b}}\frac{\partial^2}{\partial\xi^a\partial\bar{\xi^b}} \exp\left[\frac{-sz\bar{z} - tu\bar{u}}{1 + u\bar{u} + z\bar{z}}\right]$$

$$= \left(\left[\frac{s(1 + u\bar{u} - 2z\bar{z}) + t(1 + z\bar{z} - 2u\bar{u})}{1 + u\bar{u} + z\bar{z}}\right] + \left[\frac{-s^2z\bar{z}(1 + u\bar{u}) - t^2u\bar{u}(1 + z\bar{z}) + 2stu\bar{u}z\bar{z}}{(1 + u\bar{u} + z\bar{z})^2}\right]\right) \exp\left[\frac{-sz\bar{z} - tu\bar{u}}{1 + u\bar{u} + z\bar{z}}\right]$$

which gives the function $f_1(z,u)$ as

$$f_1(z,u) = \frac{-s(1+u\bar{u}-2z\bar{z}) - t(1+z\bar{z}-2u\bar{u})}{1+u\bar{u}+z\bar{z}} + \frac{s^2 z\bar{z}(1+u\bar{u}) + t^2 u\bar{u}(1+z\bar{z}) - 2stu\bar{u}z\bar{z}}{(1+u\bar{u}+z\bar{z})^2}$$

The excited states form supersymmetry quadruplets which consist of a state composed of one of the zero energy states multiplied by a scalar function plus its three supersymmetry partners. States formed from products of functions with the different zero energy states are in separate supersymmetry quadruplets.

Apart from the zero energy solution, eigensolutions of H_1 cannot be found exactly.

A more tractable form of the Hamiltonian for the purposes of perturbation theory can be obtained by using the zero energy eigensolutions as an integrating factor. With t = 0 for the moment, put

$$\Phi(z,u) = \chi(z,u)\exp\left[\frac{-sz\bar{z}}{1+u\bar{u}+z\bar{z}}\right]$$

$$H_1^{t=0}\Phi(z,u) = \left[-g^{a\bar{b}}\frac{\partial^2}{\partial\xi^a\partial\bar{\xi^b}} + f_1^{t=0}(z,u)\right]\chi(z,u)\exp\left[\frac{-sz\bar{z}}{1+u\bar{u}+z\bar{z}}\right]$$

$$= -g^{a\bar{b}}\left[\frac{\partial^2\chi(z,u)}{\partial\xi^a\partial\bar{\xi^b}} - s\left(\frac{\partial}{\partial\xi^a}\left[\frac{z\bar{z}}{1+u\bar{u}+z\bar{z}}\right]\frac{\partial\chi(z,u)}{\partial\bar{\xi^b}}\right) - s\left(\frac{\partial}{\partial\bar{\xi^b}}\left[\frac{z\bar{z}}{1+u\bar{u}+z\bar{z}}\right]\frac{\partial\chi(z,u)}{\partial\xi^a}\right)\right]\exp\left[\frac{-sz\bar{z}}{1+u\bar{u}+z\bar{z}}\right]$$

The conjugated form of the Hamiltonian

$$\hat{H}_1^{t=0} = \exp\left[\frac{sz\bar{z}}{1+u\bar{u}+z\bar{z}}\right]H_1^{t=0}\exp\left[\frac{-sz\bar{z}}{1+u\bar{u}+z\bar{z}}\right]$$

is therefore

$$\begin{aligned}\hat{H}_1^{t=0} &= -g^{a\bar{b}}\frac{\partial^2}{\partial\xi^a\partial\bar{\xi^b}} + s(1+u\bar{u}+z\bar{z})\left[\frac{(1+z\bar{z})(1+u\bar{u})}{(1+u\bar{u}+z\bar{z})^2}\left(\bar{z}\frac{\partial}{\partial\bar{z}} + z\frac{\partial}{\partial z}\right)\right.\\ &\left. - \frac{u\bar{u}z\bar{z}}{(1+u\bar{u}+z\bar{z})^2}\left(\bar{z}\frac{\partial}{\partial\bar{z}} + z\frac{\partial}{\partial z}\right) - \frac{z\bar{z}(1+u\bar{u})}{(1+u\bar{u}+z\bar{z})^2}\left(u\frac{\partial}{\partial u} + \bar{u}\frac{\partial}{\partial\bar{u}}\right) + \frac{z\bar{z}(1+u\bar{u})}{(1+u\bar{u}+z\bar{z})^2}\left(u\frac{\partial}{\partial u} + \bar{u}\frac{\partial}{\partial\bar{u}}\right)\right]\\ &= -g^{a\bar{b}}\frac{\partial^2}{\partial\xi^a\partial\bar{\xi^b}} + s\left(\bar{z}\frac{\partial}{\partial\bar{z}} + z\frac{\partial}{\partial z}\right) \quad . \end{aligned} \tag{1}$$

Due to the linearity of the derivatives in the potential term, the conjugated form of the Hamiltonian when $t \neq 0$ follows immediately

$$\hat{H}_1 = -g^{a\bar{b}}\frac{\partial^2}{\partial\xi^a\partial\bar{\xi^b}} + s\left(\bar{z}\frac{\partial}{\partial\bar{z}} + z\frac{\partial}{\partial z}\right) + t\left(\bar{u}\frac{\partial}{\partial\bar{u}} + u\frac{\partial}{\partial u}\right)$$

The general form of the Hamiltonian $\hat{H}_1$, after the substitution of a zero energy solution, valid when acting on a p-form of arbitrary p is the sum of the Laplacian and the Lie Derivative along the vector appearing in the Hamiltonian when acting on scalars, $\mathcal{L}_{s(\bar{z}\frac{\partial}{\partial\bar{z}}+z\frac{\partial}{\partial z})+t(\bar{u}\frac{\partial}{\partial\bar{u}}+u\frac{\partial}{\partial u})}$.

The form of the Hamiltonian $\hat{H}_2$ acting on excited states corresponding to the other zero energy state $\Phi_2^o(z,u)$ follows straightforwardly by transforming the parameters s and t as follows: $s\longrightarrow -s$, $t\longrightarrow -t$.

Generically the potential term in the Hamiltonian $s\left(\bar{z}\frac{\partial}{\partial\bar{z}}+z\frac{\partial}{\partial z}\right)+t\left(\bar{u}\frac{\partial}{\partial\bar{u}}+u\frac{\partial}{\partial u}\right)$ commutes with the Cartan Sub-Algebra of SU(3), I_3 and U_3, but not with any of the other generators. The SU(3) symmetry of the Hamiltonian is broken to $U(1)\times U(1)$.

For the special cases s = 0, t = 0 and s = t where the Killing Vector is a hypercharge operator, the potential term commutes with the hypercharge operator and an SU(2) subgroup of SU(3), so the SU(3) symmetry is broken to $SU(2)\times U(1)$.

By a slight rearrangement the potential term can be rewritten in terms of ladder operators and states in the SU(3) octet.

$$
\begin{aligned}
z\frac{\partial}{\partial z}+\bar{z}\frac{\partial}{\partial\bar{z}} &= \frac{1+u\bar{u}+z\bar{z}}{1+u\bar{u}+z\bar{z}}\left(z\frac{\partial}{\partial z}+\bar{z}\frac{\partial}{\partial\bar{z}}\right)\\
&= \frac{1}{1+u\bar{u}+z\bar{z}}\left(z(\frac{\partial}{\partial z}+\bar{z}^2\frac{\partial}{\partial\bar{z}})+\bar{z}(\frac{\partial}{\partial\bar{z}}+z^2\frac{\partial}{\partial z})+u\bar{u}(z\frac{\partial}{\partial z}+\bar{z}\frac{\partial}{\partial\bar{z}})\right)\\
&= \frac{1}{1+u\bar{u}+z\bar{z}}\left(z(\frac{\partial}{\partial z}+\bar{z}^2\frac{\partial}{\partial\bar{z}}+\bar{u}\bar{z}\frac{\partial}{\partial\bar{u}})+\bar{z}(\frac{\partial}{\partial\bar{z}}+z^2\frac{\partial}{\partial z}+uz\frac{\partial}{\partial u})+u\bar{z}(\bar{u}\frac{\partial}{\partial\bar{z}}-z\frac{\partial}{\partial u})+\bar{u}z(u\frac{\partial}{\partial\bar{z}}-\bar{z}\frac{\partial}{\partial\bar{u}})\right]\\
&= \chi_1^{1,0}(z,u)I_- - \chi_1^{-1,0}(z,u)I_+ - \chi_1^{-1,-1}(z,u)V_+ + \chi_1^{1,1}(z,u)V_-
\end{aligned}
$$

transforming $z\longleftrightarrow u$

$$
u\frac{\partial}{\partial u}+\bar{u}\frac{\partial}{\partial\bar{u}} = \chi_1^{0,-1}(z,u)U_+ - \chi_1^{0,1}(z,u)U_- - \chi_1^{1,1}(z,u)V_- + \chi_1^{-1,-1}(z,u)V_+
$$

The general Killing Vector leads to the term

$$
s\big(z\frac{\partial}{\partial z}+\bar{z}\frac{\partial}{\partial\bar{z}}\big)+t\big(u\frac{\partial}{\partial u}+\bar{u}\frac{\partial}{\partial\bar{u}}\big) = s\big(\chi_1^{1,0}I_- - \chi_1^{-1,0}I_+\big) - t\big(\chi_1^{0,1}U_- - \chi_1^{0,-1}U_+\big) + (s-t)\big(\chi_1^{1,1}V_- - \chi_1^{-1,-1}V_+\big)
$$

which shows that when computing corrections to the energy in perturbation theory only matrix elements between states of the same weight $i=i', j=j'$, in the same representation or one immediately above or immediately below, $n=n'$ or $n=n'\pm 1$, need to be considered.

5.5 Perturbation Theory

5.5.1 1st order perturbation theory (The Octet)

Unlike on the 2-sphere, the first order perturbative contributions on $\mathbb{CP}^2$ don't all vanish. On the 2-sphere there is an argument due to symmetry of the Hamiltonian under $z \longrightarrow \frac{1}{z}$, where z is the coordinate on the sphere, which shows that the matrix elements for first order perturbation theory vanish. However on $\mathbb{CP}^2$ no such symmetry exists.

From the form of the perturbation which is made up of first order derivatives, it is obvious that the zero energy ground state solutions which correspond to $\chi_0^{0,0}(z,u) = 1$, receive no contributions to their energy in any order of perturbation theory. The first order energy shift of the other states can be calculated by acting on the wavefunction with the perturbing term and then expanding the result as a linear combination of the eigensolutions. For the octet

$$\begin{aligned}
(z\frac{\partial}{\partial z} + \bar{z}\frac{\partial}{\partial \bar{z}})\frac{\bar{u}z}{1+u\bar{u}+z\bar{z}} &= \bar{u}z\frac{(1+u\bar{u}-z\bar{z})}{(1+u\bar{u}+z\bar{z})^2} \\
&= \frac{\bar{u}z}{(1+u\bar{u}+z\bar{z})^2}\left[\frac{1}{5}(1+u\bar{u}+z\bar{z}) + \frac{6}{5}(2-z\bar{z}) - \frac{4}{5}(2-u\bar{u})\right] \\
&= \frac{1}{5}\chi_1^{1,1} + \frac{6}{5}\chi_2^{1,1(1)} - \frac{4}{5}\chi_2^{1,1(2)}
\end{aligned}$$

$$\begin{aligned}
(z\frac{\partial}{\partial z} + \bar{z}\frac{\partial}{\partial \bar{z}})\frac{z}{1+u\bar{u}+z\bar{z}} &= z\frac{(1+u\bar{u}-z\bar{z})}{(1+u\bar{u}+z\bar{z})^2} \\
&= \frac{z}{(1+u\bar{u}+z\bar{z})^2}\left[\frac{1}{5}(1+u\bar{u}+z\bar{z}) + \frac{6}{5}(1-z\bar{z}) - \frac{2}{5}(1-2u\bar{u})\right] \\
&= \frac{1}{5}\chi_1^{1,0} + \frac{6}{5}\chi_2^{1,0(1)} - \frac{2}{5}\chi_2^{1,0(2)}
\end{aligned}$$

$$\begin{aligned}
(z\frac{\partial}{\partial z} + \bar{z}\frac{\partial}{\partial \bar{z}})\frac{\bar{u}}{1+u\bar{u}+z\bar{z}} &= \frac{-2\bar{u}z\bar{z}}{(1+u\bar{u}+z\bar{z})^2} \\
&= \frac{\bar{u}}{(1+u\bar{u}+z\bar{z})^2}\left[-\frac{2}{5}(1+u\bar{u}+z\bar{z}) - \frac{2}{5}(1-u\bar{u}) + \frac{4}{5}(1-2z\bar{z})\right] \\
&= -\frac{2}{5}\chi_1^{0,1} - \frac{2}{5}\chi_2^{0,1(2)} + \frac{4}{5}\chi_2^{0,1(1)}
\end{aligned}$$

$$\begin{aligned}
(z\frac{\partial}{\partial z} + \bar{z}\frac{\partial}{\partial \bar{z}})\frac{1-u\bar{u}}{1+u\bar{u}+z\bar{z}} &= \frac{-2z\bar{z}(1-u\bar{u})}{(1+u\bar{u}+z\bar{z})^2} \\
= -\frac{2}{5}\left[\frac{(1-u\bar{u})}{(1+u\bar{u}+z\bar{z})} - \left[\frac{\frac{3}{2}(1-z\bar{z})^2 - \frac{1}{2}(1+z\bar{z})^2}{(1+u\bar{u}+z\bar{z})^2}\right] + \left[\frac{\frac{3}{2}(u\bar{u}-z\bar{z})^2 - \frac{1}{2}(u\bar{u}+z\bar{z})^2}{(1+u\bar{u}+z\bar{z})^2}\right]\right]& \\
= -\frac{2}{5}\chi_1^{0,0(2)} + \frac{2}{5}\chi_2^{0,0(1)} - \frac{2}{5}\chi_2^{0,0(3)}&
\end{aligned}$$

$$
\begin{aligned}
(z\frac{\partial}{\partial z}+\bar{z}\frac{\partial}{\partial \bar{z}})\frac{1-z\bar{z}}{1+u\bar{u}+z\bar{z}} &= \frac{-2z\bar{z}(2+u\bar{u})}{(1+u\bar{u}+z\bar{z})^2} \\
&= \frac{1}{2}+\frac{2}{5}\frac{(1-z\bar{z})}{(1+u\bar{u}+z\bar{z})}-\frac{2}{5}\frac{(1-u\bar{u})}{(1+u\bar{u}+z\bar{z})}-\frac{3}{20}\left[\frac{\frac{3}{2}(1-u\bar{u})^2-\frac{1}{2}(1+u\bar{u})^2}{(1+u\bar{u}+z\bar{z})^2}\right] \\
&\quad +\frac{13}{20}\left[\frac{\frac{3}{2}(1-z\bar{z})^2-\frac{1}{2}(1+z\bar{z})^2}{(1+u\bar{u}+z\bar{z})^2}\right]+\frac{1}{4}\left[\frac{\frac{3}{2}(u\bar{u}-z\bar{z})^2-\frac{1}{2}(u\bar{u}+z\bar{z})^2}{(1+u\bar{u}+z\bar{z})^2}\right] \\
&= -\frac{1}{2}+\frac{2}{5}\chi_1^{0,0(1)}-\frac{2}{5}\chi_1^{0,0(2)}-\frac{3}{20}\chi_2^{0,0(1)}+\frac{13}{20}\chi_2^{0,0(2)}+\frac{1}{4}\chi_2^{0,0(3)}
\end{aligned}
$$

Degenerate perturbation theory must be used to find the first order shift for the two states with $n=1, i=0, j=0$.

$$
det\begin{bmatrix} <\chi_1^{0,0(1)}|z\frac{\partial}{\partial z}+\bar{z}\frac{\partial}{\partial\bar{z}}|\chi_1^{0,0(1)}> -\triangle^{(1)}E_1^{0,0} & <\chi_1^{0,0(2)}|z\frac{\partial}{\partial z}+\bar{z}\frac{\partial}{\partial\bar{z}}|\chi_1^{0,0(1)}> \\ <\chi_1^{0,0(1)}|z\frac{\partial}{\partial z}+\bar{z}\frac{\partial}{\partial\bar{z}}|\chi_1^{0,0(2)}> & <\chi_1^{0,0(2)}|z\frac{\partial}{\partial z}+\bar{z}\frac{\partial}{\partial\bar{z}}|\chi_1^{0,0(2)}> -\triangle^{(1)}E_1^{0,0} \end{bmatrix} = 0
$$

Substituting in for the matrix elements gives

$$
det\begin{bmatrix} \frac{2}{5}-\triangle^{(1)}E_1^{0,0} & -\frac{2}{5} \\ 0 & -\frac{2}{5}-\triangle^{(1)}E_1^{0,0} \end{bmatrix} = 0 \text{ , i.e. } \quad [\triangle^{(1)}E_1^{0,0}+\tfrac{2}{5}][\triangle^{(1)}E_1^{0,0}-\tfrac{2}{5}]=0
$$

$\triangle^{(1)}E_1^{0,0}=\frac{2}{5}$ corresponds to the eigensolution

$$
\chi_+^{0,0}=\chi_1^{0,0(1)}=\frac{1-z\bar{z}}{1+u\bar{u}+z\bar{z}}
$$

$\triangle^{(1)}E_1^{0,0}=-\frac{2}{5}$ corresponds to the eigensolution

$$
\chi_-^{0,0}=\chi_1^{0,0(1)}+2\chi_1^{0,0(2)}=\frac{3-z\bar{z}-u\bar{u}}{1+u\bar{u}+z\bar{z}} \quad .
$$

Collecting together the results for each member of the octet, the first order shifts due to the introduction of the perturbation $\triangle \hat{H} = s(z\frac{\partial}{\partial z} + \bar{z}\frac{\partial}{\partial \bar{z}})$ are

$$
\begin{array}{ccccccc}
\triangle^{(1)}E & & -\frac{2}{5}s & & \frac{1}{5}s & & \\
& & & -\frac{2}{5}s & & & \\
& \frac{1}{5}s & & & & \frac{1}{5}s & \\
& & & +\frac{2}{5}s & & & \\
& & \frac{1}{5}s & & -\frac{2}{5}s & &
\end{array}
$$

where the position of the energy shift in the diagram corresponds to the position of the eigensolution in the octet. As expected this perturbation breaks the original SU(3) symmetry to $SU(2) \times U(1)$, the SU(2) being U_{spin} which commutes with $z\frac{\partial}{\partial z} + \bar{z}\frac{\partial}{\partial \bar{z}}$, giving a triplet, two doublets and a singlet under this group.

Using the symmetry under $u \longleftrightarrow z$, the energy shifts due to the perturbation $\triangle \hat{H} = t(u\frac{\partial}{\partial u} + \bar{u}\frac{\partial}{\partial \bar{u}})$ follow immediately by a $\frac{\pi}{3}$ rotation of the previous octet diagram. This gives

$\triangle \hat{H}_1 = t(u\frac{\partial}{\partial u} + \bar{u}\frac{\partial}{\partial \bar{u}})$

$$
\begin{array}{ccccccc}
\triangle^{(1)}E & & \frac{1}{5}t & & \frac{1}{5}t & & \\
& & & -\frac{2}{5}t & & & \\
& -\frac{2}{5}t & & & & -\frac{2}{5}t & \\
& & & +\frac{2}{5}t & & & \\
& & \frac{1}{5}t & & \frac{1}{5}t & &
\end{array}
$$

Due to the linearity of the first order matrix elements, in the case of the general perturbing term $\triangle \hat{H}_1 = s(z\frac{\partial}{\partial z} + \bar{z}\frac{\partial}{\partial \bar{z}}) + t(u\frac{\partial}{\partial u} + \bar{u}\frac{\partial}{\partial \bar{u}})$, the first order energy shift is the sum of that from the two terms separately, except for the case of the two states with $i = j = 0$ where degenerate perturbation theory must be used. In this case the energy shifts correspond to the eigenvalues of the matrix

$$\begin{pmatrix} <\chi_1^{0,0(1)}|s(z\frac{\partial}{\partial z}+\bar{z}\frac{\partial}{\partial \bar{z}})+t(u\frac{\partial}{\partial u}+\bar{u}\frac{\partial}{\partial \bar{u}})|\chi_1^{0,0(1)}> & <\chi_1^{0,0(2)}|s(z\frac{\partial}{\partial z}+\bar{z}\frac{\partial}{\partial \bar{z}})+t(u\frac{\partial}{\partial u}+\bar{u}\frac{\partial}{\partial \bar{u}})|\chi_1^{0,0(1)}> \\ <\chi_1^{0,0(1)}|s(z\frac{\partial}{\partial z}+\bar{z}\frac{\partial}{\partial \bar{z}})+t(u\frac{\partial}{\partial u}+\bar{u}\frac{\partial}{\partial \bar{u}})|\chi_1^{0,0(2)}> & <\chi_1^{0,0(2)}|s(z\frac{\partial}{\partial z}+\bar{z}\frac{\partial}{\partial \bar{z}})+t(u\frac{\partial}{\partial u}+\bar{u}\frac{\partial}{\partial \bar{u}})|\chi_1^{0,0(2)}> \end{pmatrix}$$

substituting in for the matrix elements leads to

$$det\begin{bmatrix} \frac{2}{5}s-\frac{2}{5}t-\triangle^{(1)}E_1^{0,0} & -\frac{2}{5}s \\ -\frac{2}{5}t & -\frac{2}{5}s+\frac{2}{5}t-\triangle^{(1)}E_1^{0,0} \end{bmatrix}=0$$

$$[\triangle^{(1)}E_1^{0,0}]^2-[\tfrac{2}{5}]^2(s-t)^2-[\tfrac{2}{5}]^2st=0$$

which gives as the eigenvalues and their corresponding eigensolutions:

$$\triangle^{(1)}E_1^{0,0}=\tfrac{2}{5}\sqrt{(s-t)^2+st}\ , \qquad \chi_+^{0,0}=\left[\frac{s}{(s-t)-\sqrt{(s-t)^2+st}}\right]\chi_1^{0,0(1)}+\chi_1^{0,0(2)}$$

$$\triangle^{(1)}E_1^{0,0}=-\tfrac{2}{5}\sqrt{(s-t)^2+st}\ , \qquad \chi_-^{0,0}=\left[\frac{s}{(s-t)-\sqrt{(s-t)^2+st}}\right]\chi_1^{0,0(1)}-\chi_1^{0,0(2)}$$

The first order energy shifts due to the most general perturbation for each of the members of the octet corresponding to the zero energy ground state $\Phi_1^o(z,u)$ consist of

$$\triangle\hat{H}_1=s(z\tfrac{\partial}{\partial z}+\bar{z}\tfrac{\partial}{\partial\bar{z}})+t(u\tfrac{\partial}{\partial u}+\bar{u}\tfrac{\partial}{\partial\bar{u}})$$

$\triangle^{(1)}E$		$\frac{1}{5}(t-2s)$		$\frac{1}{5}(s+t)$
	$\frac{1}{5}(s-2t)$	$\pm\frac{2}{5}\sqrt{(s-t)^2+st}$	$\frac{1}{5}(s-2t)$	
		$\frac{1}{5}(s+t)$		$\frac{1}{5}(t-2s)$

which shows that only in the three cases s = 0, t = 0 and s = t is there an SU(2) subgroup unbroken.
The shift in the octet corresponding to ground state $\Phi_2^o(z,u)$ may be found by transforming $s\rightarrow -s, t\rightarrow -t$.

5.5.2 First order perturbation theory for other representations

Taking the Killing Vector $is(z\frac{\partial}{\partial z} - \bar{z}\frac{\partial}{\partial \bar{z}})$, to illustrate the effect of the perturbation on the energies of higher energy representations, the effect on the representations corresponds to $\Phi_1^o(z,u)$ is related to the effect on the representations corresponding to $\Phi_2^o(z,u)$ by changing s to –s. The first order shift in energy of the highest weight wavefunction of any of the regular hexagon representations can be found quite easily.

$$\left(z\frac{\partial}{\partial z} + \bar{z}\frac{\partial}{\partial \bar{z}}\right)\frac{\bar{u}^n z^n}{(1+u\bar{u}+z\bar{z})^n} = n\bar{u}^n z^n \frac{(1+u\bar{u}-z\bar{z})}{(1+u\bar{u}+z\bar{z})^{n+1}}$$

$$= \frac{n}{2n+3}\frac{\bar{u}^n z^n}{(1+u\bar{u}+z\bar{z})^n} + \frac{2n(n+3)}{2n+3}\frac{\bar{u}^n z^n(1-z\bar{z})}{(1+u\bar{u}+z\bar{z})^{n+1}} - \frac{2n}{2n+3}\frac{\bar{u}^n z^n(1-(n+1)u\bar{u})}{(1+u\bar{u}+z\bar{z})^{n+1}}$$

$$= \frac{n}{2n+3}\chi_n^{n,n} + \frac{2n(n+3)}{2n+3}\chi_{n+1}^{n,n(1)} - \frac{2n}{2n+3}\chi_{n+1}^{n,n(2)}$$

so that the first order shift for the highest weight states are $\triangle^{(1)}E_n^{n,n} = \left[\frac{n}{2n+3}\right]s$.

The same calculation for the state $\chi_n^{0,n}$ gives

$$\left(z\frac{\partial}{\partial z} + \bar{z}\frac{\partial}{\partial \bar{z}}\right)\frac{\bar{u}^n}{(1+u\bar{u}+z\bar{z})^n} = \frac{-2nz\bar{z}\bar{u}^n}{(1+u\bar{u}+z\bar{z})^{n+1}}$$

$$= \frac{-2n}{2n+3}\frac{\bar{u}^n}{(1+u\bar{u}+z\bar{z})^n} - \frac{2n(n+3)}{2n+3}\frac{\bar{u}^n(1-u\bar{u})}{(1+u\bar{u}+z\bar{z})^{n+1}} + \frac{n}{2n+3}\frac{\bar{u}^n(1-(n+1)z\bar{z})}{(1+u\bar{u}+z\bar{z})^{n+1}}$$

$$= \frac{-2n}{2n+3}\chi_n^{0,n} + \frac{2n}{2n+3}\chi_{n+1}^{0,n(2)} + \frac{n}{2n+3}\chi_{n+1}^{0,n(1)}$$

so that the first order energy shift for these states is $\triangle^{(1)}E_n^{0,n} = \left[\frac{-2n}{2n+3}\right]s$.

The first order corrections for the whole of the second excited regular hexagon states, the 27, can be calculated by using the fact that the $U_{spin}SU(2)$ subgroup is unbroken, that the sum of the energy shifts in first order degenerate perturbation theory is zero and that the sum of the first order energy shifts of the whole 27 must be zero. The shifts are:

$\triangle\hat{H}_1 = s(z\frac{\partial}{\partial z} + \bar{z}\frac{\partial}{\partial \bar{z}})$

$\triangle^{(1)}E_2$ $\quad -\frac{4}{7}s \quad -\frac{1}{7}s \quad \frac{2}{7}s$

$-\frac{1}{7}s \quad \mp\frac{4}{7}s \quad \mp\frac{1}{7}s \quad \frac{2}{7}s$

$\frac{2}{7}s \quad \mp\frac{1}{7}s \quad \mp\frac{4}{7}s \quad \mp\frac{1}{7}s \quad \frac{2}{7}s$

0

$\frac{2}{7}s \quad \mp\frac{1}{7}s \quad \mp\frac{4}{7}s \quad -\frac{1}{7}s$

$\frac{2}{7}s \quad -\frac{1}{7}s \quad -\frac{4}{7}s$

Taking into account that changing $z\frac{\partial}{\partial z} - \bar{z}\frac{\partial}{\partial \bar{z}}$ to $u\frac{\partial}{\partial u} - \bar{u}\frac{\partial}{\partial \bar{u}}$ rotates the diagram through $\frac{\pi}{3}$ and ensuring the correct symmetry breaking in the cases s = 0, t = 0 and s = t leads to the energy shift diagram for the general perturbation $\triangle\hat{H}_1 = s(z\frac{\partial}{\partial z} + \bar{z}\frac{\partial}{\partial \bar{z}}) + t(u\frac{\partial}{\partial u} + \bar{u}\frac{\partial}{\partial \bar{u}})$. The first order energy shifts for the 27 corresponding to the ground state $\Phi_1(z,u)$ are:

$\triangle\hat{H}_1 = s(z\frac{\partial}{\partial z} + \bar{z}\frac{\partial}{\partial \bar{z}}) + t(u\frac{\partial}{\partial u} + \bar{u}\frac{\partial}{\partial \bar{u}}), \qquad \triangle^{(1)}E_2$

$-\frac{2}{7}(2s-t) \quad \frac{1}{7}(2t-s) \quad \frac{2}{7}(s+t)$

$-\frac{1}{7}(s+t) \quad \mp\frac{1}{7}\sqrt{16s^2+t^2-16st} \quad \mp\frac{1}{7}\sqrt{s^2+t^2+14st} \quad \frac{1}{7}(2s-t)$

$\frac{2}{7}(s-2t) \quad \mp\frac{1}{7}\sqrt{s^2+16t^2-16st} \quad \mp\frac{4}{7}\sqrt{(s-t)^2+st} \quad \mp\frac{1}{7}\sqrt{s^2+16t^2-16st} \quad \frac{2}{7}(s-2t)$

0

$\frac{1}{7}(2s-t) \quad \mp\frac{1}{7}\sqrt{s^2+t^2+14st} \quad \mp\frac{1}{7}\sqrt{16s^2+t^2-16st} \quad -\frac{1}{7}(s+t)$

$\frac{2}{7}(s+t) \quad \frac{1}{7}(2t-s) \quad -\frac{2}{7}(2s-t)$

1st Order Perturbation Theory (The Decuplet)

The general form for the perturbation produced by the Killing Vector $k = is(z\frac{\partial}{\partial z} - \bar{z}\frac{\partial}{\partial \bar{z}})$ is the Lie Derivative $s\mathcal{L}_{z\frac{\partial}{\partial z}+\bar{z}\frac{\partial}{\partial \bar{z}}}$. Acting with this operator on the (0,1)-form highest weight of the decuplet

$$\psi_1^{0,2} = \frac{\bar{u}^2 dz}{(1+u\bar{u}+z\bar{z})^2}$$

gives

$$\mathcal{L}_{z\frac{\partial}{\partial z}+\bar{z}\frac{\partial}{\partial \bar{z}}}\psi_1^{0,2} = \left[di_{z\frac{\partial}{\partial z}+\bar{z}\frac{\partial}{\partial \bar{z}}} + i_{z\frac{\partial}{\partial z}+\bar{z}\frac{\partial}{\partial \bar{z}}}d\right]\psi_1^{0,2}$$
$$= \frac{\bar{u}^2 dz}{(1+u\bar{u}+z\bar{z})^2} - \frac{4\bar{u}^2 z\bar{z} dz}{(1+u\bar{u}+z\bar{z})^3}$$

which can be decomposed in terms of (1,0)-form eigensolutions as:

$$\mathcal{L}_{z\frac{\partial}{\partial z}+\bar{z}\frac{\partial}{\partial \bar{z}}}\psi_1^{0,2} = \frac{1}{2}\partial\chi_2^{0,2} + \frac{5}{6}\psi_2^{0,2(1)} - \frac{2}{3}\psi_2^{0,2(2)}$$

where

$$\psi_2^{0,2(1)} = \frac{\bar{u}^2(3(1-z\bar{z})-u\bar{u})dz}{(1+u\bar{u}+z\bar{z})^3} + \frac{2\bar{u}^3 z du}{(1+u\bar{u}+z\bar{z})^3}$$

and

$$\psi_2^{0,2(2)} = \frac{\bar{u}^2(3-2u\bar{u})dz}{(1+u\bar{u}+z\bar{z})^3} + \frac{\bar{u}^3 z du}{(1+u\bar{u}+z\bar{z})^3}$$

are members of the (1,0)-form $\mathbf{35}$ representation, and

$$\partial\chi_2^{0,2} = \frac{\bar{u}^2(1+u\bar{u}-z\bar{z})dz}{(1+u\bar{u}+z\bar{z})^3} - \frac{2\bar{u}^3 z du}{(1+u\bar{u}+z\bar{z})^3}$$

is a member of the (1,0)-form $\mathbf{27}$ representation.

Thus the contribution to the energy of the highest weight of the decuplet from first order perturbation theory vanishes. Using the Killing Vector $k = i(u\frac{\partial}{\partial u} - \bar{u}\frac{\partial}{\partial \bar{u}})$ the action of the perturbation on the highest weight state of the decuplet is:

$$
\begin{aligned}
\mathcal{L}_{u\frac{\partial}{\partial u}+\bar{u}\frac{\partial}{\partial \bar{u}}}\psi_1^{0,2} &= \frac{2\bar{u}^2 dz}{(1+u\bar{u}+z\bar{z})^2} - \frac{4\bar{u}^3 u dz}{(1+u\bar{u}+z\bar{z})^3} \\
&= -\frac{2}{3}\psi_2^{0,2(1)} + \frac{4}{3}\psi_2^{0,2(2)}
\end{aligned}
$$

The highest weight therefore has a zero contribution to its energy in first order perturbation theory for any Killing Vector. The same is in fact true for all members of the decuplet.

5.5.3 Second Order Perturbation Theory

The simplest Killing Vector to use in an example of second order perturbation theory is $k = is(z\frac{\partial}{\partial z} - \bar{z}\frac{\partial}{\partial \bar{z}} + u\frac{\partial}{\partial u} - \bar{u}\frac{\partial}{\partial \bar{u}})$ due to its symmetry under the interchange $z \leftrightarrow u$. The corresponding potential term in the Hamiltonian

$$
T = (z\frac{\partial}{\partial z} + \bar{z}\frac{\partial}{\partial \bar{z}} + u\frac{\partial}{\partial u} + \bar{u}\frac{\partial}{\partial \bar{u}}) = (\chi_1^{1,0} I_- - \chi_1^{-1,0} I_+ - \chi_1^{0,1} U_- + \chi_1^{0,-1} U_+)
$$

conserves i and j, and changes n by zero, plus one or minus one. The number of matrix elements contributing in second order perturbation theory is therefore very small, so the second order contributions to the energy of the octet corresponding to the ground state $\Phi_1^o(z,u)$, can be straightforwardly calculated.

For example the contribution to the energy of the highest weight state $\chi_1^{1,1} = \dfrac{\bar{u}z}{1+u\bar{u}+z\bar{z}}$ is

$$
\begin{aligned}
\triangle^{(2)} E_1^{1,1} &= \frac{s^2 < \chi_1^{1,1}|T|\chi_2^{1,1(1)} >< \chi_2^{1,1(1)}|T|\chi_1^{1,1} >}{(E_1 - E_2) < \chi_1^{1,1}|\chi_1^{1,1} >< \chi_2^{1,1(1)}|\chi_2^{1,1(1)} >} + \frac{s^2 < \chi_1^{1,1}|T|\chi_2^{1,1(2)} >< \chi_2^{1,1(2)}|T|\chi_1^{1,1} >}{(E_1 - E_2) < \chi_1^{1,1}|\chi_1^{1,1} >< \chi_2^{1,1(2)}|\chi_2^{1,1(2)} >} \\
&= \frac{2s^2 < \chi_1^{1,1}|T|\chi_2^{1,1(1)} >< \chi_2^{1,1(1)}|T|\chi_1^{1,1} >}{(E_1 - E_2) < \chi_1^{1,1}|\chi_1^{1,1} >< \chi_2^{1,1(1)}|\chi_2^{1,1(1)} >}
\end{aligned}
$$

the two terms being equal due to the symmetry under $z \leftrightarrow u$. By the decomposition of the operator T acting on $\chi_1^{1,1}$ used in the first order perturbation theory

$$
\frac{< \chi_2^{1,1(1)}|T|\chi_1^{1,1} >}{< \chi_2^{1,1(1)}|\chi_2^{1,1(1)} >} = \frac{2}{5}
$$

The simplest way to calculate the other matrix element is to use explicit integration over $\mathbb{CP}^2$. The volume of $\mathbb{CP}^2$ is

$$
\begin{aligned}
vol(\mathbb{CP}^2) &= \int_{\mathbb{CP}^2} \frac{1}{2}\omega \wedge \omega \\
&= \int \frac{dz d\bar{z} du d\bar{u}}{(1+u\bar{u}+z\bar{z})^3} \quad .
\end{aligned}
$$

Changing coordinates to $z = r_1 e^{i\phi_1}$, $u = r_2 e^{i\phi_2}$

$$vol(\mathbb{CP}^2) = 4 \int_0^{\infty} \int_0^{2\pi} \frac{r_1 r_2 dr_1 dr_2 d\phi_1 d\phi_2}{(1 + r_1^2 + r_2^2)^3} \quad .$$

Due to the fact that T conserves i and j any dependence on ϕ_1 and ϕ_2 in the matrix elements cancels, so the integration measure is

$$d(vol(\mathbb{CP}^2)) = \frac{16\pi^2 r_1 r_2 dr_1 dr_2}{(1 + r_1^2 + r_2^2)^3}$$

The factor of $16\pi^2$ will be neglected as it will cancel when the wavefunctions are normalized.
The action of the operator T on the wavefunction $\chi_2^{1,1(1)}$ is

$$\begin{aligned} T\chi_2^{1,1(1)} &= \left(z\frac{\partial}{\partial z} + \bar{z}\frac{\partial}{\partial \bar{z}} + u\frac{\partial}{\partial u} + \bar{u}\frac{\partial}{\partial \bar{u}} \right) \frac{\bar{u}z(2 - z\bar{z})}{(1 + u\bar{u} + z\bar{z})^2} \\ &= \frac{4\bar{u}z(1 - u\bar{u} - 2z\bar{z})}{(1 + u\bar{u} + z\bar{z})^3} \end{aligned}$$

so that in terms of the coordinates r_1 and r_2 the matrix element is

$$< \chi_1^{1,1} | T | \chi_2^{1,1(1)} >= 4 \int_0^{\infty} \frac{r_1^3 r_2^3 (1 - r_1^2 - 2r_2^2) dr_1 dr_2}{(1 + r_1^2 + r_2^2)^7} = \frac{-1}{180}$$

and the normalization term

$$< \chi_1^{1,1} | \chi_1^{1,1} >= \int_0^{\infty} \frac{r_1^3 r_2^3 dr_1 dr_2}{(1 + r_1^2 + r_2^2)^5} = \frac{1}{96} \quad .$$

Substituting each of these factors, along with $E_1 - E_2 = 3 - 8 = -5$, into the formula for the second order energy shift gives

$$\triangle^{(2)} E_1^{1,1} = 2s^2 \left(-\frac{1}{5} \right) \left(-\frac{1}{180} \right) \left(96 \right) \left(\frac{2}{5} \right) = \frac{32}{375} s^2$$

Performing the same calculation for the state $\chi_1^{1,0} = \dfrac{z}{1 + u\bar{u} + z\bar{z}}$ gives the matrix elements as

$$\frac{< \chi_2^{1,0(1)} | T | \chi_1^{1,0} >}{< \chi_2^{1,0(1)} | \chi_2^{1,0(1)} >} = \frac{4}{5} \qquad , \qquad \frac{< \chi_1^{1,0} | T | \chi_2^{1,0(1)} >}{< \chi_1^{1,0} | \chi_1^{1,0} >} = -\frac{8}{15}$$

$$\frac{<\chi_2^{1,0(2)}|T|\chi_1^{1,0}>}{<\chi_2^{1,0(2)}|\chi_2^{1,0(2)}>} = \frac{2}{5} \qquad , \qquad \frac{<\chi_1^{1,0}|T|\chi_2^{1,0(2)}>}{<\chi_1^{1,0}|\chi_1^{1,0}>} = -\frac{8}{15}$$

which gives the second order contribution to the energy

$$\triangle^{(2)} E_1^{1,0} = s^2 \left(-\frac{1}{5}\right)\left(-\frac{8}{15}\right)\left(\frac{6}{5}\right) = \frac{48}{375} s^2 \qquad .$$

By symmetry under $z \leftrightarrow u$ this is the same as for the state $\chi_1^{0,1}$,

$$\triangle^{(2)} E_1^{0,1} = \frac{48}{375} s^2 \qquad .$$

In order to calculate the second order energy shift of the states at the centre of the octet with i = j = 0, the diagonalized eigenfunctions from the first order calculation are used. These eigenfunctions are

$$\chi_+^{0,0} = \chi_1^{0,0(1)} + \chi_1^{0,0(2)} = \frac{2 - z\bar{z} - u\bar{u}}{1 + u\bar{u} + z\bar{z}}$$
$$\chi_-^{0,0} = \chi_1^{0,0(1)} - \chi_1^{0,0(2)} = \frac{z\bar{z} - u\bar{u}}{1 + u\bar{u} + z\bar{z}}$$

Acting on these functions with the operator T and decomposing the result in terms of the eigenfunctions of the Laplacian gives

$$\begin{aligned} T\chi_+ &= (z\tfrac{\partial}{\partial z} + \bar{z}\tfrac{\partial}{\partial \bar{z}} + u\tfrac{\partial}{\partial u} + \bar{u}\tfrac{\partial}{\partial \bar{u}})\chi_+ \\ &= -1 - \frac{2}{5}\chi_+ + \frac{9}{10}\chi_2^{0,0(1)} + \frac{9}{10}\chi_2^{0,0(2)} - \frac{3}{10}\chi_2^{0,0(3)} \\ T\chi_- &= (z\tfrac{\partial}{\partial z} + \bar{z}\tfrac{\partial}{\partial \bar{z}} + u\tfrac{\partial}{\partial u} + \bar{u}\tfrac{\partial}{\partial \bar{u}})\chi_- \\ &= \frac{2}{5}(\chi_- + \chi_2^{0,0(1)} - \chi_2^{0,0(2)}) \qquad . \end{aligned}$$

These expressions determine half the matrix elements needed to compute the second order contribution to the energy. After explicit integration of $\mathbb{CP}^2$ the values of the other matrix elements are

$$<\chi_+|T|\chi_2^{0,0(1)}> = <\chi_+|T|\chi_2^{0,0(2)}> = -\frac{1}{15}$$
$$<\chi_-|T|\chi_2^{0,0(1)}> = - <\chi_-|T|\chi_2^{0,0(2)}> = -\frac{1}{45}$$

and the factors due to the normalization of $\chi_\pm$ are

$$< \chi_+|\chi_+ >=< \chi_-|\chi_- >= \frac{1}{24} \qquad .$$

Combining all these results gives finally

$$\triangle^{(2)}E_{\chi_+} = \frac{s^2 < \chi_+|T|\chi_2^{0,0(1)} >< \chi_2^{0,0(1)}|T|\chi_+ >}{(E_1 - E_2) < \chi_+|\chi_+ >< \chi_2^{0,0(1)}|\chi_2^{0,0(1)} >} + \frac{s^2 < \chi_+|T|\chi_2^{0,0(2)} >< \chi_2^{0,0(2)}|T|\chi_+ >}{(E_1 - E_2) < \chi_+|\chi_+ >< \chi_2^{0,0(2)}|\chi_2^{0,0(2)} >}$$

$$= 2s^2\left(\frac{-1}{5}\right)\left(\frac{-1}{15}\right)\left(24\right)\left(\frac{9}{10}\right) = \frac{72}{125}s^2$$

$$\triangle^{(2)}E_{\chi_-} = \frac{s^2 < \chi_-|T|\chi_2^{0,0(1)} >< \chi_2^{0,0(1)}|T|\chi_- >}{(E_1 - E_2) < \chi_-|\chi_- >< \chi_2^{0,0(1)}|\chi_2^{0,0(1)} >} + \frac{s^2 < \chi_-|T|\chi_2^{0,0(2)} >< \chi_2^{0,0(2)}|T|\chi_- >}{(E_1 - E_2) < \chi_-|\chi_- >< \chi_2^{0,0(2)}|\chi_2^{0,0(2)} >}$$

$$= s^2\left(\frac{-1}{5}\right)\left(\frac{-1}{45}\right)\left(24\right)\left(\frac{2}{5}\right) + s^2\left(\frac{-1}{5}\right)\left(\frac{1}{45}\right)\left(24\right)\left(\frac{-2}{5}\right)$$

$$= \frac{32}{375}s^2 \qquad .$$

The second order contribution to the energy due to the Killing Vector $is(z\frac{\partial}{\partial z} - \bar{z}\frac{\partial}{\partial \bar{z}} + u\frac{\partial}{\partial u} - \bar{u}\frac{\partial}{\partial \bar{u}})$, for the whole octet are:

$$\frac{48}{375}s^2 \qquad \frac{32}{375}s^2$$

$$\frac{48}{375}s^2 \qquad \begin{matrix} \frac{32}{375}s^2 \\ \frac{216}{375}s^2 \end{matrix} \qquad \frac{48}{375}s^2$$

$$\frac{32}{375}s^2 \qquad \frac{48}{375}s^2$$

The previous calculation could have been performed in the same way to find the second order energy shift of the decuplet but would have been even more tedious with having to use the Lie Derivative of the perturbing vector and because there are up to four terms contributing each time as opposed to two in the case of the octet.

5.6 Asymptotic Solutions on $\mathbb{CP}^2$

5.6.1 The Harmonic Oscillator Approximation

(Isolated Fixed Points)

In the case where the Killing Vector has three isolated fixed points the Hamiltonian tends to a harmonic oscillator around each of the fixed points in the large s, large t limit. There is one zero energy eigensolution localized around each fixed point. The zero energy eigensolution localized around the first fixed point $x_1 = (1,0,0)$, or in inhomogeneous coordinates the origin in the first patch $(z=0, u=0)$ is

$$\tilde{\Psi}_1(z,u) = \exp(-sz\bar{z} - tu\bar{u})(1 - idz \wedge d\bar{z} - idu \wedge d\bar{u} + dz \wedge du \wedge d\bar{z} \wedge d\bar{u})$$

The fact that this solution has zero energy can be demonstrated by operating on it with ∂_s .

$$\partial\tilde{\Psi}_1(z,u) = \exp(-sz\bar{z} - tu\bar{u})(-s\bar{z}dz - t\bar{u}du + is\bar{z}dz \wedge du \wedge d\bar{u} - it\bar{u}dz \wedge du \wedge d\bar{z})$$
$$-i_{is\bar{z}\frac{\partial}{\partial\bar{z}} + it\bar{u}\frac{\partial}{\partial\bar{u}}}\tilde{\Psi}_1(z,u) = \exp(-sz\bar{z} - tu\bar{u})(s\bar{z}dz + t\bar{u}du - is\bar{z}dz \wedge du \wedge d\bar{u} + it\bar{u}dz \wedge du \wedge d\bar{z})$$

which gives $\partial_s\tilde{\Psi}_1(z,u) = 0$. Taking the complex conjugate it follows immediately that $\bar{\partial}_s\tilde{\Psi}_1(z,u) = 0$ as well and so $d_s\tilde{\Psi}_1(z,u) = 0$. Using the locally Euclidean metric near the fixed point, $\tilde{\Psi}_1(z,u)$ is self-dual under the Hodge Star and so is also annihilated by the conjugate operator $\bar{\delta}_s$ in the harmonic operator approximation, therefore $\tilde{\Psi}_1(z,u)$ is a zero energy eigensolution of the approximate Hamiltonian.

Around the other two fixed points, $x_2 = (0,1,0)$ or $(x=0, y=0)$ in the second patch, $x_3 = (0,0,1)$ or $(w=0, v=0)$ in the third patch, the zero energy solutions are completely analogous.
Taking $s > t \gg 0$

$$\tilde{\Psi}_2(z,u) = \exp(-sx\bar{x} - (s-t)y\bar{y})(1 + idx \wedge d\bar{x} + idy \wedge d\bar{y} + dx \wedge dy \wedge d\bar{x} \wedge d\bar{y})$$
$$\tilde{\Psi}_3(z,u) = \exp(-(s-t)v\bar{v} - tw\bar{w})(1 - idv \wedge d\bar{v} + idw \wedge d\bar{w} - dv \wedge dw \wedge d\bar{v} \wedge d\bar{w})$$

which can be seen to be annihilated by the Hamiltonian in the harmonic oscillator approximation using the form of the Killing Vector on each patch.
Topological invariants of $\mathbb{CP}^2$ are now determinable in terms of the zero energy eigensolutions localized around the fixed point set, as in section 3. There is an even-form zero energy eigensolution localized around each of the three fixed points which indicates, due to the independence of the index of the supersymmetry operators under changes in values of the parameters s and t, that the Euler Characteristic of $\mathbb{CP}^2$ is $\chi(\mathbb{CP}^2) = 3$.
It is straightforward to find the $O(s), O(t)$ excited states. Near the fixed point $x_1 = (1,0,0)$ as well as the ground state $\tilde{\Psi}_1(z,u)$ there are higher energy states

$$\tilde{\Psi}_1^t(z,u) = \exp(-sz\bar{z} - tu\bar{u})(1 - idz\wedge d\bar{z})(1 + idu\wedge d\bar{u}) \qquad , \qquad E = 4t$$
$$\tilde{\Psi}_1^s(z,u) = \exp(-sz\bar{z} - tu\bar{u})(1 + idz\wedge d\bar{z})(1 - idu\wedge d\bar{u}) \qquad , \qquad E = 4s$$
$$\tilde{\Psi}_1^{s,t}(z,u) = \exp(-sz\bar{z} - tu\bar{u})(1 + idz\wedge d\bar{z})(1 + idu\wedge d\bar{u}) \qquad , \qquad E = 4(s+t)$$

Higher energy states may be found by acting on these states with the harmonic oscillator ladder operators

$$X_+^z = \frac{\partial}{\partial \bar{z}} - sz\,, \qquad X_+^{\bar{z}} = \frac{\partial}{\partial z} - s\bar{z}\,, \qquad X_+^u = \frac{\partial}{\partial \bar{u}} - tu\,, \qquad X_+^{\bar{u}} = \frac{\partial}{\partial u} - t\bar{u}$$

The first two operators raise the energy by 2s, the second two by 2t.

The excited states form representations of $SU(2)\times SU(2)$, due to the symmetry of the harmonic oscillator Hamiltonian, of dimension $n_s n_t$ where n_s and n_t are integers. After taking into consideration the $O(1)$ contributions to the energies the energy levels will be split to form representations of $U(1)\times U(1)$, the symmetry group of the exact Hamiltonian.

Each of these states is in a separate supersymmetry quadruplet, the other three members of which may be found by applying the two supersymmetry operators in this approximation.

The energy levels of the excited states around the other fixed points are similar, but with the parameters altered due to the different forms of the Killing Vector in each of the patches. Near the second fixed point, $x_2 = (0,1,0)$, the energies of the states may be found by transforming $s \leftrightarrow -s, t \leftrightarrow s - t$. Near the third fixed point, $x_3 = (0,0,1)$, the energy may be found by transforming those in the first case with $s \leftrightarrow s - t, t \leftrightarrow t$.

5.6.2 The Harmonic Oscillator Approximation (Fixed $\mathbb{CP}^1$ Submanifold)

When the Killing Vector leaves fixed a $\mathbb{CP}^1$ submanifold the Hamiltonian asymptotically takes the form of a two-dimensional harmonic oscillator in the two directions transverse to the fixed $\mathbb{CP}^1$ plus the Laplacian on the fixed $\mathbb{CP}^1$. Taking $k = is(z\frac{\partial}{\partial z} - \bar{z}\frac{\partial}{\partial \bar{z}})$ the zero energy solutions are

$$\tilde{\Psi}_1(z,u) = \exp\left[\frac{-sz\bar{z}}{1+u\bar{u}}\right]\left[1 - i\frac{dz\wedge d\bar{z}}{(1+u\bar{u})} - i\frac{du\wedge d\bar{u}}{(1+u\bar{u})^2} + \frac{dz\wedge du\wedge d\bar{z}\wedge d\bar{u}}{(1+u\bar{u})^3}\right]$$

$$= \exp\left[\frac{-sz\bar{z}}{1+u\bar{u}}\right]\left[1 - i\frac{dz\wedge d\bar{z}}{(1+u\bar{u})}\right]\left[1 - i\frac{du\wedge d\bar{u}}{(1+u\bar{u})^2}\right]$$

$$\tilde{\Psi}_3(z,u) = \exp\left[\frac{-sz\bar{z}}{1+u\bar{u}}\right]\left[1 - i\frac{dz\wedge d\bar{z}}{(1+u\bar{u})} + i\frac{du\wedge d\bar{u}}{(1+u\bar{u})^2} - \frac{dz\wedge du\wedge d\bar{z}\wedge d\bar{u}}{(1+u\bar{u})^3}\right]$$

$$= \exp\left[\frac{-sz\bar{z}}{1+u\bar{u}}\right]\left[1 - i\frac{dz\wedge d\bar{z}}{(1+u\bar{u})}\right]\left[1 + i\frac{du\wedge d\bar{u}}{(1+u\bar{u})^2}\right]$$

which are localized around the fixed $\mathbb{CP}^1$. These solutions are the product of the transverse harmonic oscillator ground state with the self and anti-self-dual representatives of the cohomology of the fixed $\mathbb{CP}^1$. The other approximate zero energy solutions are localized around the fixed point $x_2 = (0,1,0)$, the origin in the second patch

$$\tilde{\Psi}_2(z,u) = \exp[-sx\bar{x} - sy\bar{y}](1 + idx\wedge d\bar{x} + idy\wedge d\bar{y} + dx\wedge dy\wedge d\bar{x}\wedge d\bar{y}) \qquad .$$

These three zero energy even-form eigensolutions again illustrate the Lefschetz Fixed Point Theorem, giving $\chi(\mathbb{CP}^1) = 3$.

In this case, i.e. t = 0, the Hamiltonian on scalar excitations of $\Psi_1(z,u)$ and $\Psi_2(z,u)$ is

$$H_1 = \triangle + f_1^{t=0}(z,u)$$

where $\triangle$ is the Laplacian on scalars and $f_1^{t=0}(z,u)$ is

$$f_1^{t=0}(z,u) = -s\left(\frac{1+u\bar{u}-2z\bar{z}}{1+u\bar{u}+z\bar{z}}\right) + s^2\left(\frac{(1+u\bar{u})z\bar{z}}{(1+u\bar{u}+z\bar{z})^2}\right)$$

Substituting $w = \sqrt{s}z$ and retaining terms of order $O(s)$, so that for large s, near the fixed $\mathbb{CP}^1(z=0,u)$, H_1 takes the approximate form of a harmonic oscillator

$$\tilde{H}_1 = -(1+u\bar{u})s\frac{d^2}{dwd\bar{w}} + s\frac{w\bar{w}}{1+u\bar{u}} - s$$

The solutions being

$$\tilde{\Psi}_1^{p,q}(z,u) = \left[\frac{d}{d\bar{z}} - \frac{sz}{1+u\bar{u}}\right]^p\left[\frac{d}{dz} - \frac{s\bar{z}}{1+u\bar{u}}\right]^q \exp\left(-\frac{-sz\bar{z}}{1+u\bar{u}}\right)$$

with energy $E = (p+q)s$ and $i = p - q$ where i is the eigenvalue of the central charge $\mathcal{L}_{z\frac{\partial}{\partial z} - \bar{z}\frac{\partial}{\partial\bar{z}}}$.

5.6.3 The Hirzebruch Signature

The Hirzebruch Signature $\tau(M)$ of $\mathbb{CP}^2$, the number of self-dual forms minus the number of anti-self-dual forms, is one. It is always possible to put the eigensolutions of the Laplacian in a basis where they are all either self or anti-self-dual by pairing eigensolutions with their dual under the Hodge Star. In this basis every non-zero energy self-dual eigensolution is paired with an anti-self-dual solution, so only zero energy solutions contribute to the signature. On $\mathbb{CP}^2$ in this basis the zero energy eigensolutions of the Laplacian are,

$$1+\frac{1}{2}\omega\wedge\omega \quad \text{, which is self-dual}$$
$$1-\frac{1}{2}\omega\wedge\omega \quad \text{, which is anti-self-dual}$$
$$\omega \quad \text{, which is self-dual.}$$

After introducing the Killing Vector into the supersymmetry algebra the index of $d_s+\delta_s$ acting on $\Lambda^{\pm}(M)$ must still be equal to the signature for all values of s and t.

The relationship between the Killing Vector and the signature can be seen in terms of the form of the Killing Vector on the three different patches.

$$\tau(M)=\sum_j(-1)^{\theta_j}$$

where θ_j is the number of the parameters λ_i which are negative in the Killing Vector on the patch j. On $\mathbb{CP}^2$ the Killing Vector with $s>t>0$,

$$k=is\left(z\frac{\partial}{\partial z}-\bar{z}\frac{\partial}{\partial\bar{z}}\right)+it\left(u\frac{\partial}{\partial u}-\bar{u}\frac{\partial}{\partial\bar{u}}\right)$$

has no negative λ_i on the first patch, two negative on the second patch and one negative on the third, so $\tau(\mathbb{CP}^2)=1$, see section 5.3.

Alternatively, in terms of the orientation on each patch,

$$\tau(M)=\sum_j n_j$$

where $n_j=+1$ or -1 depending on whether or not the orientation on the patch agrees with the natural orientation on $\mathbb{CP}^2$ after the Killing Vector has been put in the standard form. On the first and second patches the orientation is the same as the natural orientation and on the third patch it is opposite.

When s and t are large $s \gg t \gg 0$ and the low energy solutions become localized around the fixed point, the zero energy solutions around the fixed points in the first and second patches are self-dual and that around the fixed point in the third patch is anti-self-dual, as in section 5.6.1.

When there is a fixed $\mathbb{CP}^1$ e.g. t = 0 and s large one of the zero energy solutions around the fixed $\mathbb{CP}^1$ is self-dual and the other is anti-self-dual, because $\tau(\mathbb{CP}^1) = 0$, and the zero energy solutions localized around the fixed point is self-dual, as in the previous section, again confirming $\tau(\mathbb{CP}^2) = 1$.

5.6.4 $O(1)$ Corrections

The $O(1)$ corrections to the energy break the symmetry from $SU(2) \times SU(2)$, where one $SU(2)$ is the isometry group of a fixed $\mathbb{CP}^1$ and the other is due to the symmetry of the transverse harmonic oscillator, to $SU(2) \times U(1)$ the symmetry group of the exact Hamiltonian.

To find the $O(1)$ contributions to the asymptotic expansion for the energy it is easiest to use the form of the Hamiltonian obtained after using $\Phi_1^o(z,u)$ as an integrating factor, equation (1)

$$\hat{H}_1 = \triangle + s(z\tfrac{\partial}{\partial z} + \bar{z}\tfrac{\partial}{\partial \bar{z}})$$

Substituting $w = \sqrt{s}z$ into this equation and retaining terms of order $O(1)$ gives

$$\hat{H} = -[s(1+u\bar{u}) + 2w\bar{w} + u\bar{u}w\bar{w}]\frac{\partial^2}{\partial w \partial \bar{w}} - (1+u\bar{u})^2\frac{\partial^2}{\partial u \partial \bar{u}} - (1+u\bar{u})[\bar{u}w\frac{\partial^2}{\partial w \partial \bar{u}} + u\bar{w}\frac{\partial^2}{\partial u \partial \bar{w}}] + s[w\frac{\partial}{\partial w} + \bar{w}\frac{\partial}{\partial \bar{w}}]$$

The lowest energy eigensolutions of $\hat{H}$ are the Associated Legendre Functions

$$\tilde{\Psi}_1^{0,m}(z,u) = Y_l^m(u) \quad , \qquad E = l(l+1) \quad , \qquad i = 0, j = m \quad .$$

The next lowest energy eigensolutions take the form of expansions in powers of $(1+u\bar{u})^{-1}$, i.e.

$$\tilde{\Psi}_{1n}^{1,0}(z,u) = \sum_{k=1}^{n} \frac{a_k w}{(1+u\bar{u})^k} \quad , \qquad i = 1, j = 0 \quad .$$

Inserting this expansion into the Schrödinger Equation gives the relation

$$a_{k+1} = \left[\frac{s + k^2 - E}{k(k+1)}\right] a_k$$

so that for closed solutions $E = s + k^2$, where k is an integer greater than or equal to zero. These solutions are each in an SU(2) multiplet under the unbroken U_{spin} subgroup of SU(3). The other members of the multiplets may be found by using the ladder operators U_+, U_-. Taking complex conjugates gives another set of multiplets with $i = -1$.

The first excited state of the Laplacian on $\mathbb{CP}^2$, the octet with energy E = 3, has split in the large s limit to form an SU(2) triplet

$$Y_1^{-1}(u) = \frac{u}{1+u\bar{u}} \quad , \quad Y_1^0(u) = \frac{1-u\bar{u}}{1+u\bar{u}} \quad , \quad Y_1^1(u) = \frac{\bar{u}}{1+u\bar{u}} \quad .$$

with energy E = 2,

two SU(2) doublets

$$\tilde{\chi}^{-1,0}(z,u) = \frac{\bar{z}}{1+u\bar{u}} \quad , \quad \tilde{\chi}^{-1,-1}(z,u) = \frac{u\bar{z}}{1+u\bar{u}}$$
$$\tilde{\chi}^{1,1}(z,u) = \frac{\bar{u}z}{1+u\bar{u}} \quad , \quad \tilde{\chi}^{0,1}(z,u) = \frac{z}{1+u\bar{u}}$$

with energy $E = s + 1$,

and a singlet

$$\tilde{\chi}^{0,0}(z,u) = \frac{z\bar{z}}{1+u\bar{u}} + \frac{1}{s(1-2s)}$$

with energy $E = 2s - 1$.

5.7 Summary

The ordinary Laplacian on $\mathbb{CP}^2$ is invariant under the isometry group SU(3). Which representations are formed by the eigensolutions of the Laplacian is determined by the Frobenius Reciprocity Theorem.

After the introduction of a Killing Vector of the general form $\mathbf{k} = is(z\frac{\partial}{\partial z} - \bar{z}\frac{\partial}{\partial \bar{z}}) + it(u\frac{\partial}{\partial u} - \bar{u}\frac{\partial}{\partial \bar{u}})$ into the supersymmetry algebra the Hamiltonian is symmetrical under $U(1) \times U(1)$ in the general case and $SU(2) \times U(1)$ when the Killing Vector leaves fixed a $\mathbb{CP}^1$ submanifold. A formula of Witten's involving the cohomology of the fixed submanifold has been used to find three families of closed but not exact even-forms in the sense of d_s. This corresponds with the fact that the Euler Characteristic of $\mathbb{CP}^2$ is $\chi(\mathbb{CP}^2) = 3$. The zero energy eigensolutions of the Laplacian corresponding to d_s are more difficult to find; the two self-dual ones have been found, but unfortunately the anti-self-dual one hasn't/

The change in energy of the excited states was calculated using perturbation theory and used to illustrate the breaking of the SU(3) symmetry.

In the large s, large t limit the low energy solutions become localized around the fixed submanifold of the Killing Vector. In the generic case when the Killing Vector just leaves three points fixed the Hamiltonian takes the approximate form of a four-dimensional harmonic oscillator around each fixed point. There is one zero energy even-form solution around each fixed point again corresponding via the Lefschetz Fixed Point Theorem to $\chi(\mathbb{CP}^2) = 3$.

Two zero energy solutions are self-dual and one is anti-self-dual corresponding to $\tau(\mathbb{CP}^2) = 1$.

When the fixed point set consists of a fixed $\mathbb{CP}^1$ and a fixed point, in the harmonic oscillator approximation the zero energy solutions consist of two around the fixed $\mathbb{CP}^1$ with one self-dual, one anti-self-dual and one around the fixed point which is self-dual.

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